Involution matrix loci and orbit harmonics

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May 29, 2025

Published in: Mathematische Zeitschrift

DOI: 10.1007/s00209-025-03736-4

Orbit harmonics

Orbit harmonics is a practical method in combinatorial representation theory. Given a finite locus $\mathcal{Z} \in \mathbb{C}^N$, orbit harmonics yields a graded finite $\mathbb{C}$-algebra $R(\mathcal{Z})$.

- $\mathbb{C}[\mathcal{Z}] \cong R(\mathcal{Z})$ as an isomorphism of vector spaces.
- If there exists a finite subgroup $G \subset GL_N(\mathbb{C})$ acting on $\mathcal{Z}$, then $\mathbb{C}[\mathcal{Z}] \cong R(\mathcal{Z})$ as an isomorphism of G-modules.

Orbit harmonics provides an algebraic version $R(\mathcal{Z})$ for a combinatorial object $\mathcal{Z}$. Furthermore, $R(\mathcal{Z})$ is a graded refinement of $\mathbb{C}[\mathcal{Z}]$. We are interested in the representation theory and combinatorics on them.

Former works about orbit harmonics on finite matrix loci

Rhoades [2024] initiated the application of orbit harmonics to finite matrix loci $\mathcal{Z} \subset \mathrm{Mat}_{n\times n}(\mathbb{C})$. He considered the locus $\mathfrak{S}_n \subset \mathrm{Mat}_{n\times n}(\mathbb{C})$ of permutation matrices carrying an action of the group $\mathfrak{S}_n \times \mathfrak{S}_n \subset GL(\mathrm{Mat}_{n\times n}(\mathbb{C}))$ by left and right multiplication. Liu [2024] extended this work to colored permutations.

Our work

In our work, we consider two classes of involution matrix loci in $\mathrm{Mat}_{n\times n}(\mathbb{C})$ as follows. Both carry the conjugate $\mathfrak{S}_n$-action $g \cdot z := gzg^{-1}$.

- $\mathcal{M}_n := \{w \in \mathfrak{S}_n \,:\, w^2 = 1\} \subset \mathrm{Mat}_{n\times n}(\mathbb{C})$
- $\mathcal{M}_{n,a} := \{w \in \mathcal{M}_n \,:\, w \text{ has exactly } a \text{ fixed points}\}$

The following things are calculated explicitly by us.

- The graded $\mathfrak{S}_n$-module structure of $R(\mathcal{M}_n)$ and $R(\mathcal{M}_{n,a})$.
- An explicit linear basis of $R(\mathcal{M}_n)$.
- An explicit ring presentation of $R(\mathcal{M}_n)$ and $R(\mathcal{M}_{n,0})$

Orbit harmonics

Given a finite locus $\mathcal{Z} \subset \mathbb{C}^N$, orbit harmonics yields the graded $\mathbb{C}$-algebra $R(\mathcal{Z})$ step by step.

- The *vanishing ideal* $\mathrm{I}(\mathcal{Z}) \subseteq \mathbb{C}[x_1, \cdots, x_N]$ is given by

$$\mathrm{I}(\mathcal{Z}) := \{f \in \mathbb{C}[x_1, \cdots, x_N] \,:\, f(z) = 0 \text{ for all } z \in Z\}.$$

- For any ideal $I \subseteq \mathbb{C}[x_1, \cdots, x_N]$, the *graded ideal* of I is the homogeneous ideal

$$\mathrm{gr} I := \langle \tau(f) \,:\, f \in I, f \neq 0 \rangle$$

  where $\tau(f)$ is the highest-degree homogeneous component of $f \neq 0$.

- $R(\mathcal{Z}) := \mathbb{C}[x_1, \cdots, x_N]/\mathrm{gr}\mathrm{I}(\mathcal{Z})$.

Standard properties of orbit harmonics

Recall that $R(\mathcal{Z}) := \mathbb{C}[x_1, \cdots, x_N]/\mathrm{grI}(\mathcal{Z})$ is a graded $\mathbb{C}$-algebra. We have the following isomorphisms of vector spaces.

- $\mathbb{C}[\mathcal{Z}] \cong \mathbb{C}[x_1, \cdots, x_N]/\mathrm{I}(\mathcal{Z})$
- $\mathbb{C}[x_1, \cdots, x_N]_{\leq d}/\mathrm{I}(\mathcal{Z}) \cap \mathbb{C}[x_1, \cdots, x_N]_{\leq d} \cong R(\mathcal{Z})_{\leq d}$
- In particular, $\mathbb{C}[\mathcal{Z}] \cong \mathbb{C}[x_1, \cdots, x_N]/\mathrm{I}(\mathcal{Z}) \cong R(\mathcal{Z})$

If there exists a finite subgroup $G \subset GL_N(\mathbb{C})$ acting on $\mathcal{Z}$, then the above-stated linear isomorphisms can be replaced by isomorphisms of G-modules.

Note: The isomorphism $\mathbb{C}[x_1, \cdots, x_N]_{\leq d}/\mathrm{I}(\mathcal{Z}) \cap \mathbb{C}[x_1, \cdots, x_N]_{\leq d} \cong_G R(\mathcal{Z})_{\leq d}$ is abstract without an explicit map.

Standard Young tableaux

Definition

Let $\lambda \vdash n$ be a partition. A *standard Young tableau* of shape λ is a bijective filling of $[n] = \{1, \cdots, n\}$ into the Young diagram of shape λ, such that the entries are increasing across rows and down columns. $\mathrm{SYT}(\lambda) := \{\text{Standard Young tableaux of shape } \lambda\}$.

Example

$$n = 7, \lambda = (4, 2, 1)$$

1	3	6	7
2	5		
4			

Schensted correspondence

The Schensted correspondence [Schensted, 1961] is a bijection

$$\begin{aligned} \mathfrak{S}_n &\xrightarrow{1:1} \bigsqcup_{\lambda \vdash n} \{(P, Q) \,:\, P, Q \in \mathrm{SYT}(\lambda)\} \\ w &\longmapsto (P(w), Q(w)). \end{aligned}$$

Symmetric functions

- Write $\Lambda = \bigoplus_{n \geq 0} \Lambda_n$ for the graded algebra of symmetric functions in an infinite variable set $\{x_1, x_2, \cdots\}$ over the ground field $\mathbb{C}(q)$.
- Λ_n possesses a linear basis consisting of Schur functions $\{s_\lambda : \lambda \vdash n\}$.
- For $f, g \in \Lambda$, we can define the plethysm $f[g] \in \Lambda$.
- In particular, we have

$$s_d[s_2] = \sum_{\substack{\lambda \vdash n \\ \lambda \text{ even}}} s_\lambda.$$

Frobenius image

- Irreducible $\mathfrak{S}_n$-modules are in one-to-one correspondence with partitions of n. That is, Specht modules V^λ.
- Let $V \cong \bigoplus_{\lambda \vdash n} c_\lambda V^\lambda$ be an $\mathfrak{S}_n$-module. Then the *Frobenius image* of V is the symmetric function

$$\mathrm{Frob}(V) := \sum_{\lambda \vdash n} c_\lambda s_\lambda.$$

- If $V = \bigoplus_{d \geq 0} V_d$ is a graded $\mathfrak{S}_n$-module, we define the *graded Frobenius image* of V by

$$\mathrm{grFrob}(V; q) = \sum_{d \geq 0} \mathrm{Frob}(V_d) \cdot q^d.$$

Frobenius image

Lemma

$\mathrm{Frob}(\mathcal{M}_{n,a}) = s_{(n-a)/2}[s_2] \cdot s_a$ *if* $a \equiv n \mod 2$.

Notations

- From now on, we focus on the loci $\mathcal{M}_n$ and $\mathcal{M}_{n,a}$.
- For convenience, the orbit harmonics variable set $\{x_1, \cdots, x_N\}$ is rearranged into a matrix

$$\mathbf{x}_{n\times n} := \{x_{i,j} : 1 \leq i, j \leq n\}.$$

Generators of $\mathrm{grI}(\mathcal{M}_n)$

Proposition (Liu-Ma-Rhoades-Z.'25)

$\mathrm{grI}(\mathcal{M}_n)$ *can be generated by:*

- *all squares* $x_{i,j}^2$ *of variables,*
- *all sums* $x_{i,1} + \cdots + x_{i,n}$ *of variables in a single row,*
- *all sums* $x_{1,j} + \cdots + x_{n,j}$ *of variables in a single column,*
- *all products* $x_{i,j} \cdot x_{i,j'}$ *of variables in a single row,*
- *all products* $x_{i,j} \cdot x_{i',j}$ *of variables in a single column, and*
- *all diagonally symmetric differences* $x_{i,j} - x_{j,i}$ *of variables.*

A linear basis of $R(\mathcal{M}_n)$

Definition

Let $w \in \mathcal{M}_n$ be an involution given by $w = (i_1\, j_1) \cdots (i_m\, j_m)$ where $i_k < j_k$ for $1 \leq k \leq m$. Then we define the *matching monomial* $\mathrm{m}(w) \in \mathbb{C}[\mathbf{x}_{n\times n}]$ by

$$\mathrm{m}(w) := \prod_{k=1}^{m} x_{i_k, j_k}.$$

A linear basis of $R(\mathcal{M}_n)$

Theorem (Liu-Ma-Rhoades-Z.'25)

The set $\{\mathrm{m}(w) : w \in \mathcal{M}_n\}$ of matching monomials descends to a linear basis of $R(\mathcal{M}_n)$.

Corollary (Liu-Ma-Rhoades-Z.'25)

The Hilbert series of $R(\mathcal{M}_n)$ is given by

$$\mathrm{Hilb}(R(\mathcal{M}_n); q) = \sum_{d=0}^{\lfloor n/2 \rfloor} \binom{n}{2d} \cdot (2d-1)!! \cdot q^d.$$

The graded $\mathfrak{S}_n$-module structure of $R(\mathcal{M}_n)$

Theorem (Liu-Ma-Rhoades-Z.'25)

The graded Frobenius image of $R(\mathcal{M}_n)$ is given by

$$\mathrm{grFrob}(R(\mathcal{M}_n); q) = \sum_{k=0}^{\lfloor n/2 \rfloor} q^k \cdot s_k[s_2] \cdot s_{n-2k}.$$

Generators of $\mathrm{grI}(\mathcal{M}_{n,0})$

Proposition (Liu-Ma-Rhoades-Z.'25)

For $n > 0$ even, we have
$\mathrm{grI}(\mathcal{M}_{n,0}) = \mathrm{grI}(\mathcal{M}_n) + \langle x_{i,i} : 1 \leq i \leq n \rangle.$

The graded module structure of $R(\mathcal{M}_{n,0})$

Theorem (Liu-Ma-Rhoades-Z.'25)

For $n > 0$ even, the graded Frobenius image of $R(\mathcal{M}_{n,0})$ is given by

$$\mathrm{grFrob}(R(\mathcal{M}_{n,0}); q) = \sum_{\substack{\lambda \vdash n \\ \lambda \text{ even}}} q^{\frac{n-\lambda_1}{2}} \cdot s_\lambda.$$

Corollary (Liu-Ma-Rhoades-Z.'25)

For $n > 0$ even, the Hilbert series of $R(\mathcal{M}_{n,0})$ is given by

$$\mathrm{Hilb}(R(\mathcal{M}_{n,0}); q) = \sum_{w \in \mathcal{M}_{n,0}} q^{\frac{n-\mathrm{lds}(w)}{2}}.$$

Basis?

The graded module structure of $R(\mathcal{M}_{n,a})$

Theorem (Liu-Ma-Rhoades-Z.'25)

Suppose $a \equiv n \mod 2$. The graded Frobenius image of $R(\mathcal{M}_{n,a})$ is given by

$$\mathrm{grFrob}(R(\mathcal{M}_{n,a}); q) = \sum_{d=0}^{(n-a)/2} \{s_d[s_2] \cdot s_{n-2d} - s_{d-1}[s_2] \cdot s_{n-2d+2}\}_{\lambda_1 \leq n-2d+a} \cdot q^d$$

with the convention that $s_{-1} = 0$.

Hilbert series? Basis?

Open problems

Problem

Find a canonical linear basis for $R(\mathcal{M}_{n,a})$ *when* $a \equiv n \mod 2$.

Problem

Explicitly compute the Hilbert series $\mathrm{Hilb}(R(\mathcal{M}_{n,a}); q)$ *when* $a \equiv n \mod 2$ *and* $a > 1$.

Problem

Show that both $R(\mathcal{M}_n)$ *and* $R(\mathcal{M}_{n,a})$ *are* $\mathfrak{S}_n$*-log-concave.*

Problem

Consider other cycle types or colored permutations.

Thank you very much for your attention!

References I

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