Rational Torsion Points of Abelian Varieties

joint work with Ziyang Gao

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Torsion points

Given an elliptic curve E over a number field K. We have the following Mordell-Weil Theorem. (It also holds for abelian varieties, which we shall discuss later.)

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$E(K) \simeq \mathbb{Z}^r \oplus$ (finite abelian group).

A natural question is thus raised:

Question 1.2

What can we say about the finite abelian group $E(K)_{\mathrm{tor}}$? Can we determine all possibilities of what it could be?

Mazur's Theorem

In 1977, Mazur proved the following striking theorem:

Theorem 1.3

For $K = \mathbb{Q}$, there are only the following 15 possibilities of $E(\mathbb{Q})_{\mathrm{tor}}$:

- $\mathbb{Z}/N\mathbb{Z}$, $N = 1, 2 \cdots, 10$ *or* $N = 12$.
- $\mathbb{Z}/2 \times \mathbb{Z}/N\mathbb{Z}$, $N = 2, 4, 6, 8$.

In particular, $\#E(\mathbb{Q}) \leq 16$

Merel's result

The result was strengthened by Kamienny, Merel, Parent, for elliptic curves defined over $K \neq \mathbb{Q}$.

Theorem 1.4 (Merel, 1994)

Let $d = [K : \mathbb{Q}]$, E/K be an elliptic curve. If $E(K)$ has a p-torsion point, then $p < d^{3d^2}$.

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In this way, Parent finally deduce the Uniform Boundedness Conjecture (UBC) for elliptic curves over an arbitary number field. The following conjecture plays the main role in today's talk.

Conjecture 1.5 (Ogg, Osterlè, Mazur...)

There is a constant $C = C(K, g)$, such that $\#A(K)_{\mathrm{tor}} \leq C$ holds for any g dimensional abelian variety A/K.

Their methods

Parent's Theorem is as follows. His constant C only depends on $d = [K : \mathbb{Q}]$!

Theorem 1.6 (Parent, 2014)

There is a bound $C = C([K : \mathbb{Q}])$, such that $\#E(K)_{\mathrm{tor}} \leq C$ holds for any elliptic curve E/K.

Proof idea:

- Translate UBC to existence of rational points on modular curve $X_1(N)$.
- Use the Jacobian $J_1(N)$ of $X_1(N)$, and the mod p Galois representations.

The proof works only for elliptic curves, and cannot be generalize to higher dimensional abelian varieties. The main issue is, for higher dimensional abelian variety, the moduli space is not a curve!

The Function Field Setting

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Conjecture 2.1 (UBC for function fields)

There is a constant $C = C(K, g)$, such that for K being a function field, and A/K being a g dimensional abelian variety, we have

$$\#(A/Tr_{K/k}A)(K)_{\mathrm{tor}} \leq C(g, K).$$

Comments on UBC for function fields

- There is a slightly modification involving the K/k-trace $Tr_{K/k}A$. Loosely speaking (e.g if $\mathrm{char}\, k = 0$), it is the maximal abelian subvariety of A, which is defined over (a finite extension) of k.

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- There is a slightly modification involving the K/k-trace $Tr_{K/k}A$. Loosely speaking (e.g if $\mathrm{char}\, k = 0$), it is the maximal abelian subvariety of A, which is defined over (a finite extension) of k.
- It is necessary to quotient out $Tr_{K/k}A$, as if there is an abelian variety A_0/k such that $A = A_0 \otimes_k K$, then

$$A(K) = \mathrm{Hom}_{k(B)}(k(B), A) = \mathrm{Hom}_k(B, A_0)$$

can be infinite (for example, $k = \mathbb{C}$ and A_0 is the Jacobian of B)

Results for Function Fields

We can now formulate our main theorem. The following non-quantitative result is firstly proved by Looper-Yap'26 recently without semistable assumption.

Theorem 3.1 (Looper-Yap 26', Gao-G. 26+)

UBC holds for semistble, traceless abelian varieties over characteristic 0 function fields.

Quantitative Result

More explicitly, we proved the quantitative result.

Theorem 3.2 (Gao-G. 26+)

Let $K = \mathbb{C}(B)$ be a complex function field. Let g_0 be the genus of the curve B. Let (A, L) be a traceless, polarized abelian variety over K, so that no factor of A has good reduction everywhere. Then

$$\#A(K)_{\text{tor}} \leq h^0(A, L)\left(3 \cdot 10^7\right)^g (2g)^{g^2} (g_0 + 1)^{2g} := C_{UBC}.$$

By Zarhin's Trick, we can replace g by $8g$ to eliminate the dependence on $h^0(A, L)$. The fniteness of good reduction part is handled by Shafarevich, Zarhin, Parshin, Faltings... So this theorem in particular implies UBC.

Remarks

- Compare with earlier results of Looper-Yap, we are able to give (partially) explicit bound, depending on $g = \dim A$ and (polynomially on) the genus of base curve $g(B)$. However, the results of Looper-Yap is non-effective, but they can prove the full UBC for function field.
- Our methods both proves the Lang-Silverman Conjecture in the function field case.
- Rather than using Galois representations, these two proofs are all based on developments of height theory, or what we called *Arakelov Geometry*.
- Our proof also gives a result for number fields, in terms of *(generalized) Szpiro ratio* (which is known to be bounded in the function field case).

Lang-Silverman Conjecture

The Lang-Silverman Conjecture predicts the minimal height of a rational point on an abelian variety should be comparable with the height of the abelian variety.

Theorem 3.3 (Gao-G. 26+)

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$$\#\hat{h}_L(P) \geq \frac{1}{50000(C_{UBC}+1)^2 8^g g^{\frac{g}{2}+4} g_0^2} \max\{h_{Fal}(A), 1\}.$$

Result for Number Fields

Theorem 3.4 (Gao-G.26+)

For a semistable abelian variety A defined over a number field K,

$$\#A(K)_{\mathrm{tor}} \leq C(g, [K:\mathbb{Q}]) \cdot \sigma^{2g}.$$

Here $\sigma = \sigma_{A/K}$ is the generalized Szpiro ratio, which we shall define later.
The constant $C(g, [K:\mathbb{Q}]) \sim [K:\mathbb{Q}] \log[K:\mathbb{Q}]$ when regarded as a function of $[K:\mathbb{Q}]$.

Remark 3.5

As Arakelov Geometry enables us a unified way to tackle number fields and function fields, we can also obtain a result towards Lang-Silverman Conjecture in this case.

Szpiro Ratio

Usually function field analogues are easier to prove, and shed some lights on the proof for number fields. For example, the famous *abc* conjecture is true by elementary methods over function field. Also, its equivalent form, the Szpiro conjecture is known for function field.

Definition 4.1

For E over a number field K (say $K = \mathbb{Q}$), the Szpiro ratio is $\sigma := \frac{\log \Delta_{E/K}}{\log \mathfrak{f}_{E/K}}$.

- $\Delta_{E/K}$ is the minimal discriminant (ideal). If we are given a minimal equation $E : y^2 = x^3 + ax + b$, then $\Delta_{E/K} = 4a^3 + 27b^2$.
- $\mathfrak{f}_{E/K}$ is the conductor (ideal), which is the radical ideal of $\Delta_{E/K}$.

Szpiro conjecture and Néron components

For number fields Szpiro conjectured the quantity σ is bounded independent of E. More precisely, say for $K = \mathbb{Q}$, the conjecture is:

Conjecture 4.2 (Szpiro)

For every $\epsilon > 0$, there are only finitely many elliptic curve $E/\mathbb{Q}$ satisfies $\sigma > 6 + \epsilon$.

The key to understand it is via the Néron model. Loosely speaking, for every prime $\mathfrak{p} \nmid \infty$ of K,

$$\operatorname{ord}_{\mathfrak{p}} \Delta_{E/K} = \#\{\text{Components of the special fiber of Néron model of } E\}$$

So Szpiro conjecture provides a bound for the "averaged size" of Néron model.

Generalized Szpiro Ratio

For an abelian variety A defined over a number field or a function field, let $\mathrm{M}_{K,A}^{\mathrm{bad}}$ be the places where A has bad reduction (including archimedean places). In our paper, We define the *generalized Szpiro ratio* to be

$$\sigma_{A/K} := \sigma := \frac{h_F(A/K)}{\sum_{v \in \mathrm{M}_{K,A}^{\mathrm{bad}}} (\#\Phi_v)^{\frac{1}{r_v}}}.$$

Here, r_v is the toric rank of A_v, and Φ_v is the Néron component group of A_v.

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Here, r_v is the toric rank of A_v, and Φ_v is the Néron component group of A_v.

The main obstacle of UBC is to control Φ_v for different v, which amounts to control σ.

Proof Strategy

Note that our method is similar to Vojta's proof of Mordell conjeture via Diophantine Approximation.

- The main innovation of our method is the introduction of an adelic choice of*bump functions*. This idea comes from Arakelov Geometry. This enables us to produce extra positivity, namely *the sum of the 'local heights':= boundary height*, which is comparable with the Faltings height of A (assuming Szpiro conjecture). The idea is to cancel Faltings height, we obtain uniformity.
- We choose an *auxilary section* s of L^n so that the norm is small, and then compute height by (derivatives of) s.
- We employ the classical method, *multiplicity estimate*, to construct an *auxiliary point* x on which s does not vanish too much.
- The *height inequality* on the jet space of $L(x)$ gives the uniform result.

Thank you for your attention!