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# On the extinction-extinguishing dichotomy for a stochastic Lotka–Volterra type population dynamical system

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**Abstract**

Applying the Foster–Lyapunov type criteria and a martingale method, we study a two-dimensional process $(X, Y)$ arising as the unique nonnegative solution to a pair of stochastic differential equations driven by independent Brownian motions and compensated spectrally positive Lévy random measures. Both processes $X$ and $Y$ can be identified as continuous-state nonlinear branching processes where the evolution of $Y$ is negatively affected by $X$. Assuming that process $X$ extinguishes, i.e. it converges to 0 but never reaches 0 in finite time, and process $Y$ converges to 0, we identify rather sharp conditions under which the process $Y$ exhibits, respectively, one of the following behaviors: extinction with probability one, extinguishing with probability one or both extinction and extinguishing occurring with strictly positive probabilities.


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## 1. Introduction and main results

### 1.1. Introduction on the background, the model and the approach

Lotka–Volterra model serves as a fundamental ecological system. The deterministic Lotka–Volterra model for population dynamics describes the evolution of two species suffering from both self-regulations and interspecific competitions for limiting resources. A stochastic Lotka–Volterra process generalizes the deterministic Lotka–Volterra population dynamics to incorporate the influence of demographic stochasticity or random environmental fluctuations. In Cattiaux-Méléard [5], an interacting logistic Feller diffusion system is proposed as a stochastic Lotka–Volterra dynamics whose quasi-stationary distribution is studied. Two different spatial Lotka–Volterra type models are formulated in Blath et al. [4] as lattice-indexed interacting Feller diffusions and lattice-indexed interacting Fisher-Wright diffusions, respectively, where the persistence and long term coexistence of the populations are investigated. Evans et al. [9] consider a two-dimensional diffusion that solves a system of stochastic differential equations with Lotka–Volterra type drift and linear diffusion coefficients driven by a correlated two-dimensional Brownian motion, and study its stationary distribution. Hening and Nguyen [11] further generalize the model of Evans et al. [9] and prove results on the rate of convergence to the stationary distribution. Similar models have also been studied systematically as solution to a system of stochastic differential equations driven by both Brownian motions and Poisson random measures. We refer to Zhu and Yin [26] and Bao et al. [1] and references therein for previous work.

In the above mentioned models, the drift coefficients and (or) the diffusion coefficients are assumed to be of particular forms. Hening et al. [12] recently proposed populations dynamics described by $n$-dimensional Kolmogorov systems with nonlinear interactions and driven by white noise. Sharp conditions are found for the populations to converge exponentially fast to their stationary distributions and for the populations to converge to 0 exponentially fast. We refer to Benaïm [3] for a comprehensive study on stochastic persistence and related topics for general interacting SDE systems.

On the other hand, progress has been made on the study of continuous-state branching processes with generalized branching mechanism. The extinction, explosion and coming down from infinity results for such processes are obtained in Li et al. [20] via martingale approaches. This motivates us to further study similar behaviors for the general continuous-state branching processes with interaction.

In this paper we consider a generalized version of the stochastic competitive Lotka–Volterra process $(X, Y)$ arising as the non-negative, spectrally positive solution to a system of stochastic differential equations (SDEs for short) driven by independent Brownian motions and compensated Poisson random measures.

Intuitively, the process $X$ represents the (re-scaled) size of a population with a certain type of individuals whose evolution is described by a continuous-state branching process with a general nonadditive branching mechanism that has been studied in Li et al. [20]. We also refer to Li [18] for a review on continuous-state branching processes. Process $Y$ represents a population of another type that is a continuous-state branching process experiencing a competition pressure from $X$. From another point of view, one can also identify $X$ as the environment that affects the evolution of process $Y$.

Process $(X, Y)$ can also be treated as a generalized two-type continuous-state nonlinear branching process. The readers are referred to Ma [21] and Barczy et al. [2] for two-type continuous-state branching processes, and to Li [16], Hong and Li [13], Chapter 6 of Li [17] and the references therein for two-type measure-valued branching processes.

In the study of the Lotka–Volterra process, people are often interested in whether the two different populations still coexist in the long run, or whether there is only a mono-type population left eventually. For a continuous-state branching process people also want to distinguish between extinction and extinguishing that are two distinct ways of converging to 0 as time goes to infinity. We say extinction occurs if the process reaches 0 in finite time, and extinguishing occurs if the process converges to 0 but never reaches 0 in finite time. In this paper we want to carry out more detailed analysis of the extinction-extinguishing behaviors for process $Y$ given that it converges to 0 eventually, and want to understand how the processes $X$ and $Y$ jointly affect the extinction-extinguishing behaviors of process $Y$.

Note that in SDE terminology the above mentioned extinction and extinguishing behaviors correspond to the accessibility/inaccessibility of boundary 0 for the associated SDE. To our best knowledge, such boundary classifications are rarely known for an interacting system of SDEs with jumps.

As a first attempt of studying such interacting population dynamics under general setting, we first consider two populations that both undergo nonlinear subcritical branching. We further assume that the interaction between the two populations is one-sided, i.e. the evolution of process $Y$ is affected by process $X$ while the impact of $Y$ on $X$ is negligible. We thus propose and study the following SDE system:

$$\begin{cases} X_t = X_0 - \int_0^t a_1(X_s)\mathrm{d}s + \int_0^t a_2(X_s)^{1/2}\mathrm{d}B_s + \int_0^t \int_0^\infty \int_0^{a_3(X_{s-})} z\tilde{M}(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u), \\ Y_t = Y_0 - \int_0^t [b_1(Y_s) + \theta(Y_s)\kappa(X_s)]\mathrm{d}s + \int_0^t b_2(Y_s)^{1/2}\mathrm{d}W_s \\ \qquad + \int_0^t \int_0^\infty \int_0^{b_3(Y_{s-})} z\tilde{N}(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u), \end{cases} \tag{1.1}$$

where functions $a_i, b_i$ $(i = 1, 2, 3)$ and $\theta, \kappa$ are nonnegative functions on $[0, \infty)$, $(B_t)_{t\geq 0}$ and $(W_t)_{t\geq 0}$ are Brownian motions, $\{\tilde{M}(\mathrm{d}t, \mathrm{d}z, \mathrm{d}u)\}$ and $\{\tilde{N}(\mathrm{d}t, \mathrm{d}z, \mathrm{d}u)\}$ are compensated Poisson random measures with intensity $\mathrm{d}t\mu(\mathrm{d}z)\mathrm{d}u$ and $\mathrm{d}t\nu(\mathrm{d}z)\mathrm{d}u$, respectively, and with the $\sigma$-finite non-zero measures $\mu$ and $\nu$ satisfying

$$\int_0^\infty (z \wedge z^2)\mu(\mathrm{d}z) + \int_0^\infty (z \wedge z^2)\nu(\mathrm{d}z) < \infty.$$

We also assume that $(B_t)_{t\geq 0}$, $(W_t)_{t\geq 0}$, $\{\tilde{M}(\mathrm{d}t, \mathrm{d}z, \mathrm{d}u)\}$ and $\{\tilde{N}(\mathrm{d}t, \mathrm{d}z, \mathrm{d}u)\}$ are independent of each other.

Since (1.1) represents a stochastic continuous-state Lotka–Volterra population system, by a solution $(X, Y)$ to (1.1) we mean a càdlàg $\mathbb{R}_+^2$-valued process $(X, Y)$ that satisfies Eq. (1.1) up to the minimum of the first time of either hitting zero or explosion for both processes $X$ and $Y$, which is a variation of the usual definition of solution to SDE; see Definition 1.1. Conditions on the existence and uniqueness of the solution to (1.1) will be given in Lemma A.1. Since we are only interested in the solution up to the first time of hitting 0, the uniqueness holds under mild conditions. The uniqueness of such a solution for SDE had been studied before in Dawson et al. [8] and Li [19].

The extinction/extinguishing behaviors of the continuous-state nonlinear branching process $X$ have been studied in Li et al. [20] using a martingale approach. By imposing conditions on SDEs (1.1) so that the solution $X$ extinguishes with probability one and the solution $Y$ converges to 0 in probability as time goes to infinity, in this paper we find conditions under which the process $Y$ becomes extinct in finite time with probability one and zero, respectively. We further show that under certain conditions, both extinction and extinguishing can happen for $Y$ each with a strictly positive probability, which is a remarkable phenomena.

For stable Poisson random measures with stable indices in $(1, 2)$ and for power function coefficients in the SDEs in (1.1), the conditions can be made more explicit in terms of the powers and the stable indices, and they turn out to be quite sharp. We are not aware of similar previous results on solutions to such a system of general SDEs with jumps.

Our main approach is different from that in Li et al. [20]. To prove the above mentioned results we first develop stochastic Foster–Lyapunov type criteria with localized conditions for probability of finiteness of the first time of hitting 0 by either process $X$ or $Y$. These criteria can be compared with those in Li et al. [20] for solution to one-dimensional SDE and are of independent interest. We refer to Chen [6] and Meyn and Tweedie [23] for the (deterministic) Foster–Lyapunov type criteria for explosion and stability of Markov chains. The proofs of most of the main results then boil down to finding appropriate test functions in order to apply the stochastic Foster–Lyapunov type criteria, and the localized conditions in the Foster–Lyapunov criteria make it more convenient to construct the test functions.

It is remarkable that for the model in Li et al. [20] the Foster–Lyapunov criteria also produce very sharp results; see recent work in Ma et al. [22].

We apply the Foster–Lyapunov criteria to show most of the main results. The key is to identify the right test functions for which our approach is mostly ad hoc. We typically start with elementary functions such as power functions and exponential functions, then modify and (or) combine these functions in different ways to develop the sharpest possible results. Verification of the criteria often involve lengthy computations.

Among the main results, applying the stochastic Foster–Lyapunov criteria we identify sufficient conditions for the process $Y$ to become extinguishing with probability one or to become extinct with a strictly positive probability.

To find conditions under which the process $Y$ extinguishes with a strictly positive probability, we adopt a different approach, where by first obtaining an estimate on the time dependent lower bound of the sample paths of $X$, we apply a martingale argument similar to that in Li et al. [20] together with a comparison theorem. We also use either the stochastic Foster–Lyapunov criteria or the martingale method to study the extinction-extinguishing behaviors for some critical cases.

The rest of the paper is arranged as follows. We first present the main results together with an example of SDEs with power coefficients and stable Poisson random measures in Section 1.2. The Foster–Lyapunov type criteria are proved in Section 2. Proofs of the main results are deferred to Section 3.

### 1.2. Main results

We first present some notations and assumptions. By Taylor's formula (see (3.5) and (3.6) in Section 3 of the following), for $u, z > 0$ and $\delta \geq -1$,

$$(1+z)^{-\delta} - 1 + \delta z = \delta(\delta+1)z^2 \int_0^1 (1+zv)^{-\delta-2}(1-v)\mathrm{d}v$$

and

$$\ln(1+z) - z = \ln(1+z) - \ln 1 - z = -z^2 \int_0^1 (1+zv)^{-2}(1-v)\mathrm{d}v.$$

We also want to introduce several auxiliary functions. For $\delta \in (-1,0) \cup (0,\infty)$, and $u > 0$ define

$$\begin{aligned} H_{1,\delta}(u) &:= \frac{1}{\delta(\delta+1)} \int_0^\infty [(1+zu^{-1})^{-\delta} - 1 + \delta z u^{-1}]\mu(\mathrm{d}z) \\ &= u^{-2} \int_0^\infty z^2 \mu(\mathrm{d}z) \int_0^1 (1+zu^{-1}v)^{-2-\delta}(1-v)\mathrm{d}v, \end{aligned} \tag{1.2}$$

$$\begin{aligned} H_{2,\delta}(u) &:= \frac{1}{\delta(\delta+1)} \int_0^\infty [(1+zu^{-1})^{-\delta} - 1 + \delta z u^{-1}]\nu(\mathrm{d}z) \\ &= u^{-2} \int_0^\infty z^2 \nu(\mathrm{d}z) \int_0^1 (1+zu^{-1}v)^{-2-\delta}(1-v)\mathrm{d}v. \end{aligned} \tag{1.3}$$

For $u > 0$ let

$$\begin{aligned} H_{1,0}(u) &:= -\int_0^\infty \Big(\ln(1+zu^{-1}) - zu^{-1}\Big)\mu(\mathrm{d}z) \\ &= u^{-2} \int_0^\infty z^2 \mu(\mathrm{d}z) \int_0^1 (1+zu^{-1}v)^{-2}(1-v)\mathrm{d}v, \end{aligned} \tag{1.4}$$

$$\begin{aligned} H_{2,0}(u) &:= -\int_0^\infty \Big(\ln(1+zu^{-1}) - zu^{-1}\Big)\nu(\mathrm{d}z) \\ &= u^{-2} \int_0^\infty z^2 \nu(\mathrm{d}z) \int_0^1 (1+zu^{-1}v)^{-2}(1-v)\mathrm{d}v \end{aligned} \tag{1.5}$$

and

$$G_{1,0}(u) := a_1(u)u^{-1} + 2^{-1}a_2(u)u^{-2} + a_3(u)H_{1,0}(u), \tag{1.6}$$

$$G_{2,0}(u) := b_1(u)u^{-1} + 2^{-1}b_2(u)u^{-2} + b_3(u)H_{2,0}(u). \tag{1.7}$$

These six functions will appear repeatedly throughout the paper. The functions $H_{1,\delta}$ and $H_{2,\delta}$ are the same to the function $\delta(\delta+1)H_{\delta-1}$ defined in (2.1) of [20] which result from Ito's formula applied to power function of $X$. The functions $G_{1,0}$ and $G_{2,0}$ can be regarded as the limits of $(1-a)^{-1}G_a$ when $a \to 1$ in (2.3) of [20]. They are also associated to Ito's formula applied to logarithm function of $X$; see [22]. To study the extinction-extinguishing phenomena of $Y$ we impose some conditions on $G_{1,0}, G_{2,0}$; see Condition 1.6.

Let $C^2((0,\infty))$ be the space of twice continuously differentiable functions on $(0,\infty)$ and $C^2((0,\infty)\times(0,\infty))$ denote space of functions on $(0,\infty)\times(0,\infty)$ with continuous second partial derivatives.

For any generic stochastic process $Z := (Z(t))_{t\ge 0}$ and constant $w > 0$, let

$$\tau_0^Z := \inf\{t \ge 0 : Z(t) = 0\}, \quad \tau_w^Z := \tau^Z(w) := \inf\{t \ge 0 : Z(t) < w\} \tag{1.8}$$

and

$$\sigma_w^Z := \sigma^Z(w) := \inf\{t \ge 0 : Z(t) > w\} \tag{1.9}$$

with the convention $\inf \emptyset = \infty$. In the following we state the definition of solution to SDE (1.1), which is defined before the minimum of the first time of either hitting zero or explosion for the two processes.

**Definition 1.1.** By a solution to SDE (1.1) we mean that a two-dimensional càdlàg process $(X_t, Y_t)_{t\geq 0}$ satisfies SDE (1.1) up to $\gamma_n := \tau^X_{1/n} \wedge \tau^Y_{1/n} \wedge \sigma^X_n \wedge \sigma^Y_n$ for each $n \geq 1$ and $X_t = \limsup_{n\to\infty} X_{\gamma_n -}$ and $Y_t = \limsup_{n\to\infty} Y_{\gamma_n -}$ for $t \geq \lim_{n\to\infty} \gamma_n$.

**Remark 1.2.** The above definition of solution to SDE (1.1) allows weaker conditions for uniqueness of solution. In particular, the pathwise uniqueness holds if the functions $a_i, b_i, \theta$ and $\kappa$ are all locally Lipschitz on $(0, \infty)$; see Lemma A.1. Also observe that

$$\tau^X_0 \wedge \tau^Y_0 = \lim_{n\to\infty} \tau^X_{1/n} \wedge \tau^Y_{1/n}.$$

Throughout this paper we assume that the càdlàg $\mathbb{R}^2_+$-valued process $(X, Y)$ is the unique solution to (1.1), and consequently, the process $(X, Y)$ has the strong Markov property. We always assume that $X_0, Y_0 > 0$ and that all the stochastic processes are defined on the same filtered probability space $(\Omega, \mathscr{F}, \mathscr{F}_t, \mathbf{P})$. Let $\mathbf{E}$ be the corresponding expectation.

Throughout the paper we also assume that the following conditions hold.

(C1) The functions $a_i, b_i$ $(i = 1, 2, 3)$, $\theta$ and $\kappa$ are nonnegative and bounded on any bounded interval;

(C2) For each $c' > 0$,

$$\sup_{0<u\leq c'} [G_{1,0}(u) + G_{2,0}(u)] < \infty;$$

(C3) For each $0 < c' < c''$,

$$\inf_{x\in[c',c'']} \{a_2(x) + a_3(x)\} > 0, \quad \inf_{x\in[c',c'']} \{b_2(x) + b_3(x)\} > 0 \ \text{ and } \mu((c', c'')) > 0.$$

**Remark 1.3.** Under the above conditions, with probability one, both processes $X_t$ and $Y_t$ converge to 0 as $t \to \infty$. But $X_t$ does not become extinct almost surely by [20, Theorem 2.3 (i) and Proposition 2.6]; see also Lemma 3.2 in the following. In this situation we say extinguishing occurs for process $X$. Note that process $Y$ also becomes extinguishing under the above conditions if $\kappa \equiv 0$. The following theorems give the conditions on the extinction-extinguishing phenomena of $Y$.

We first find conditions distinguishing between extinction with probability 0 and extinction with a positive probability for process $Y$.

**Theorem 1.4.** *If there exists a constant* $c^* > 0$ *so that* $\sup_{0<u\leq c^*} \theta(u)u^{-1} < \infty$, *then* $\mathbf{P}\{\tau^Y_0 < \infty\} = 0$.

**Theorem 1.5.** *Suppose that there exist constants* $c^*, c_1 > 0$, $\theta \in [0, 1)$ *and* $\delta > 1$ *so that*

$$\inf_{c_1\leq u\leq c^*} \kappa(u) > 0, \quad \inf_{0<u\leq c^*} \theta(u)u^{-\theta} > 0, \ \textit{and} \ \inf_{0<u\leq c^*} \left[a_2(u)u^{-2-\delta} + a_3(u)u^{-\delta-1}\right] > 0.$$

*Then* $\mathbf{P}\{\tau^Y_0 < \infty\} > 0$.

From the above two theorems we find that the extinction of $Y$ is caused by $X$ through the negative drift coefficient function $-\theta(v)\kappa(u)$ for $u$ near zero, and not caused by the Brownian driven or Poisson-random-measure driven components of the SDE for $Y$. Intuitively, process $Y$ becomes extinguishing or extinct depending on whether $\theta(u)$ converges to 0 fast enough or slow enough as $u \to 0+$. Note that under the conditions on $\theta$, the role of function $\kappa$ in these theorems is not essential.

To further study the extinction-extinguishing behaviors of process $Y$ we need to introduce more sets of conditions. Since both $X$ and $Y$ have no negative jumps, we impose upper and lower power function bounds on functions $G_{i,0}(u)$, $\theta(u)$ and $\kappa(u)$ only for $u$ close to 0. These conditions help to simplify arguments in proofs and allow more transparent conditions (in terms of powers of the power functions) for the extinction-extinguishing behaviors for $Y$.

**Condition 1.6.**

(i) *There exist constants* $\theta \in [0, 1)$, $c^*, c_\theta, a, b, \kappa > 0$ *and* $p, q \geq 0$ *so that*

  (ia) $G_{1,0}(u) \leq au^p$ *for all* $0 < u \leq c^*$;
  (ib) $G_{2,0}(u) \geq bu^q$ *for all* $0 < u \leq c^*$;
  (ic) $\theta(u) \geq c_\theta u^\theta$ *and* $\kappa(u) \geq u^\kappa$ *for all* $0 < u \leq c^*$.

(ii) *There exist constants* $\theta \in [0, 1)$, $c^*, c_\theta, a, b, \kappa > 0$ *and* $p, q \geq 0$ *so that*

  (iia) $G_{1,0}(u) \geq au^p$ *for all* $0 < u \leq c^*$;
  (iib) $G_{2,0}(u) \leq bu^q$ *for all* $0 < u \leq c^*$;
  (iic) $\theta(u) \leq c_\theta u^\theta$ *and* $\kappa(u) \leq u^\kappa$ *for all* $0 < u \leq c^*$.

(iii) *Assume that the function* $u \mapsto b_3(u)$ *is nondecreasing and that the functions* $\theta, b_1, b_2, b_3$ *are locally Lipschitz, that is, for each closed interval* $[u, v] \subset (0, \infty)$, *there is a constant* $C(u, v) \geq 0$ *so that*

$$|\theta(x) - \theta(y)| + \sum_{i=1,2,3} |b_i(x) - b_i(y)| \leq C(u, v)|x - y|$$

*for all* $u \leq x, y \leq v$.

We remark that Condition 1.6(iii) is needed for a comparison theorem, Proposition 3.6, which is applied in proofs for Theorems 1.8 and 1.10.

The following theorems further distinguish between extinction with a positive probability and extinction with probability one for $Y$. In particular, we identify conditions under which both extinction and extinguishing happen with a strictly positive probability.

**Theorem 1.7.** *Suppose that Condition 1.6(i) holds with*

$$\frac{q\kappa}{q + 1 - \theta} < p. \tag{1.10}$$

*Then* $\mathbf{P}\{\tau_0^Y < \infty\} = 1$.

**Theorem 1.8.** *Suppose that Condition 1.6(ii) and (iii) hold with*

$$\frac{q\kappa}{q + 1 - \theta} > p > 0. \tag{1.11}$$

*Then* $\mathbf{P}\{\tau_0^Y < \infty\} < 1$.

In the following we consider Condition 1.6 for either $pq = 0$ or $\frac{q\kappa}{q+1-\theta} = p$ with $p, q > 0$. Observe that the case for $p > 0$ and $q = 0$ is addressed in Theorem 1.7 on the extinction behavior.

**Theorem 1.9.** *Suppose that Condition 1.6(i) holds for constants satisfying one of the following conditions:*

(i) $p = q = 0$ *and* $b/a > \kappa/(1-\theta)$,
(ii) $p, q > 0$, $\frac{q\kappa}{q+1-\theta} = p$, *and*

$$\frac{ap}{q(q+1-\theta)} < \Big(\frac{b}{1-\theta}\Big)^{\frac{1-\theta}{q+1-\theta}} \cdot \Big(\frac{c_\theta}{q}\Big)^{\frac{q}{q+1-\theta}}. \tag{1.12}$$

*Then* $\mathbf{P}\{\tau_0^Y < \infty\} = 1$.

**Theorem 1.10.** *Suppose that Condition 1.6(ii) and (iii) hold for constants satisfying one of the following conditions:*

(i) $p = q = 0$ *and* $b/a < \kappa/(1-\theta)$,
(ii) $p = 0$, $q > 0$.

*Then* $\mathbf{P}\{\tau_0^Y < \infty\} < 1$.

Given the above theorems, there are still cases for the parameters $a, b, c_\theta, p, q, \theta, \kappa$ in which the extinction-extinguishing behaviors are unknown.

**Conjecture 1.11.** *We conjecture that* $\mathbf{P}\{\tau_0^Y < \infty\} < 1$ *if Condition 1.6(ii) and (iii) hold with* $p, q > 0$, $\frac{q\kappa}{q+1-\theta} = p$ *and*

$$\frac{ap}{q(q+1-\theta)} > \Big(\frac{b}{1-\theta}\Big)^{\frac{1-\theta}{q+1-\theta}} \cdot \Big(\frac{c_\theta}{q}\Big)^{\frac{q}{q+1-\theta}}.$$

To better understand the conditions, we next consider an example of SDE system (1.1) with power function coefficients and stable Poisson random measures.

**Example 1.12.** Suppose that there are constants $a_i, b_i, \theta \geq 0$, $\kappa, \eta > 0$, $\alpha_1, \alpha_2 \in (1, 2)$ and $q_i, p_i \geq 0$ so that $\kappa(u) = u^\kappa$, $\theta(u) = \eta u^\theta$,

$$a_i(u) = a_i u^{p_i+i},\ b_i(u) = b_i u^{q_i+i} \text{ for } i = 1, 2 \text{ and } a_3(u) = a_3 u^{p_3+\alpha_1},\ b_3(u) = b_3 u^{q_3+\alpha_2},$$

and

$$\mu(\mathrm{d}z) = \frac{\alpha_1(\alpha_1-1)}{\Gamma(2-\alpha_1)} z^{-1-\alpha_1}\mathrm{d}z, \quad \nu(\mathrm{d}z) = \frac{\alpha_2(\alpha_2-1)}{\Gamma(2-\alpha_2)} z^{-1-\alpha_2}\mathrm{d}z$$

with Gamma function $\Gamma$. We also assume that $a_2 + a_3 > 0$ and $b_2 + b_3 > 0$. Then $H_{1,0}(u) = \Gamma(\alpha_1)u^{-\alpha_1}$, $H_{2,0}(u) = \Gamma(\alpha_2)u^{-\alpha_2}$ and

$$G_{1,0}(u) = a_1 u^{p_1} + 2^{-1}a_2 u^{p_2} + a_3\Gamma(\alpha_1)u^{p_3}, \quad G_{2,0}(u) = b_1 u^{q_1} + 2^{-1}b_2 u^{q_2} + b_3\Gamma(\alpha_2)u^{q_3}.$$

Let

$$p := \min\{p_1 1_{\{a_1\neq 0\}}, p_2 1_{\{a_2\neq 0\}}, p_3 1_{\{a_3\neq 0\}}\},$$
$$q := \min\{q_1 1_{\{b_1\neq 0\}}, q_2 1_{\{b_2\neq 0\}}, q_3 1_{\{b_3\neq 0\}}\}$$

and

$$a := a_1 1_{\{p_1=p\}} + \frac{a_2}{2} 1_{\{p_2=p\}} + a_3\Gamma(\alpha_1)1_{\{p_3=p\}},$$
$$b := b_1 1_{\{q_1=q\}} + \frac{b_2}{2} 1_{\{q_2=q\}} + b_3\Gamma(\alpha_2)1_{\{q_3=q\}}.$$

Note that constants $p$ and $q$ are the minimum powers of the power functions in the expressions of $G_{1,0}$ and $G_{2,0}$, respectively, and the associated power function or functions dominate the behaviors of the corresponding polynomial $G_{i,0}(u)$ for $u$ near zero. The constants $a$ and $b$ represent coefficients of the (possibly combined) dominant power functions, respectively. Since processes $X$ and $Y$ have no negative jumps, these dominant power functions together determine the extinction/extinguishing behaviors of $Y$.

Combining Theorems 1.4–1.5 and 1.7–1.10, we have

(i) $\mathbf{P}\{\tau_0^Y < \infty\} = 0$ if $\theta \geq 1$;
(ii) $\mathbf{P}\{\tau_0^Y < \infty\} > 0$ if $0 \leq \theta < 1$;
(iii) $\mathbf{P}\{\tau_0^Y < \infty\} = 1$ if $0 \leq \theta < 1$ and one of the following holds:

- (iiia) $p = q = 0$ and $b/a > \kappa/(1-\theta)$;
- (iiib) $p > 0$ and $q = 0$;
- (iiic) $p, q > 0$ and $\frac{q\kappa}{q+1-\theta} < p$;
- (iiid) $p, q > 0$, $\frac{q\kappa}{q+1-\theta} = p$ and

$$\frac{ap}{q(q+1-\theta)} < \left(\frac{b}{1-\theta}\right)^{\frac{1-\theta}{q+1-\theta}} \cdot \left(\frac{\eta}{q}\right)^{\frac{q}{q+1-\theta}};$$

(iv) $0 < \mathbf{P}\{\tau_0^Y < \infty\} < 1$ if $0 \leq \theta < 1$ and one of the following holds:

- (iva) $p = q = 0$ and $b/a < \kappa/(1-\theta)$;
- (ivb) $p = 0$ and $q > 0$;
- (ivc) $p, q > 0$ and $\frac{q\kappa}{q+1-\theta} > p$.

**Remark 1.13.** From the above example we have the following insights. The extinction of $Y$ is caused by relatively large negative interaction $-\eta Y_t^\theta X_t^\kappa$. If $\theta > 0$ is small enough, $Y_t^\theta$ decreases slowly enough as $Y_t \to 0+$ and there is enough negative drift to cause extinction.

- If $\kappa > 0$ is further relatively small, then $X_t^\kappa$ also decreases slowly enough as $X_t \to 0+$ so that there is a large enough negative drift $-\eta Y_t^\theta X_t^\kappa$ that causes extinction for $Y$ with probability one.
- On the other hand, if $\kappa > 0$ is not relatively small, then the negative drift $-\eta Y_t^\theta X_t^\kappa$ becomes small enough when $X_t$ starts to take small values. In this case, $Y$ can survive with a positive probability.

## 2. Two-dimensional stochastic Foster–Lyapunov type criteria

The study of boundary behaviors for Markov processes started with the boundary classification of Markov chains and some Chinese probabilist had made important contributions on it. Among them Mu-Fa Chen identified the explosion/non-explosion conditions for continuous time Markov chains and Markov jump processes in 1980s; see the review paper Chen [7] and the book Chen [6] and references therein where the uniqueness and non-uniqueness problems essentially correspond to the non-explosion and the explosion, respectively, for Markov chains. Khasminskii [15] proved similar conditions for diffusion processes. These conditions were later referred to as Foster and Lyapunov criteria for more general Markov process; see e.g. Meyn and Tweedie [23].

Using one-dimensional Foster–Lyapunov type criteria, an estimate is found in Section 4 of [20] on the first passage probabilities for the continuous-state nonlinear branching process $X$.

A Foster–Lyapunov type criterion is also identified in [22] for non-extinction of the continuous-state nonlinear branching process. These criteria generalize similar results for Markov chains; see Chen [6, Theorems 2.25 and 2.27]. The conditions for Propositions 2.1 and 2.2 in the following are also similar to those of [6, Theorems 2.25 and 2.27] which are the criteria of uniqueness of $q$-process.

In this section we establish the two-dimensional criteria, which will be used to prove Theorems 1.4, 1.5, 1.7 and 1.9. In the following, let $(x_t, y_t)_{t\geq 0}$ with $x_0, y_0 > 0$ denote a two-dimensional Markov process where $(x_t)_{t\geq 0}$ and $(y_t)_{t\geq 0}$ are two nonnegative processes defined before the minimum of their first times of hitting 0 or explosion. Let $L_t$ be an operator such that for each $g \in C^2((0,\infty)\times(0,\infty))$ and $m, n \geq 1$, the process $t \mapsto M^g_{t\wedge\gamma_{m,n}}$ is a local martingale, where

$$M^g_t := g(x_t, y_t) - g(x_0, y_0) - \int_0^t L_s g(x_s, y_s)\mathrm{d}s \tag{2.1}$$

and $\gamma_{m,n} := \tau_n \wedge \sigma_m$ with $\tau_n := \tau^x_{1/n} \wedge \tau^y_{1/n}$ and $\sigma_m := \sigma^x_m \wedge \sigma^y_m$. Then a natural candidate for $L_t$ is the generator of the process $(x_t, y_t)_{t\geq 0}$. Define the stopping time $\tau_0 := \tau^x_0 \wedge \tau^y_0$. Since the two processes $(x_t)_{t\geq 0}$ and $(y_t)_{t\geq 0}$ are defined before the first time of hitting zero or explosion,

$$\tau_0 = \lim_{n\to\infty} \tau_n. \tag{2.2}$$

**Proposition 2.1.** *Suppose that there is a non-negative function $g \in C^2((0,\infty)\times(0,\infty))$ and a sequence of positive constants $(d_m)_{m\geq 1}$ satisfying*

(i) $\lim_{x\wedge y\to 0+} g(x, y) = \infty$;
(ii) $L_t g(x, y) \leq d_m g(x, y)$ *for all* $t > 0$, $x, y \in (0, m)$ *and all large* $m \geq 1$.

*Then* $\mathbf{P}\{\tau_0 < \infty\} = 0$.

**Proof.** Observe that there is a sequence of stopping times $(\gamma_k)_{k\geq 1}$ so that $\gamma_k \to \infty$ almost surely as $k \to \infty$ and $t \mapsto M^g_{t\wedge\gamma_{m,n,k}}$ is a martingale for each $m, n, k \geq 1$, where $\gamma_{m,n,k} := \gamma_{m,n} \wedge \gamma_k$. By (2.1) and condition (ii), for each $m, n, k \geq 1$ and $t \geq 0$,

$$\begin{aligned}\mathbf{E}\big[g(x_{t\wedge\gamma_{m,n,k}}, y_{t\wedge\gamma_{m,n,k}})\big] &= g(x_0, y_0) + \int_0^t \mathbf{E}\big[L_s g(x_s, y_s)1_{\{s\leq\gamma_{m,n,k}\}}\big]\mathrm{d}s \\ &\leq g(x_0, y_0) + d_m \int_0^t \mathbf{E}\big[g(x_s, y_s)1_{\{s\leq\gamma_{m,n,k}\}}\big]\mathrm{d}s \\ &\leq g(x_0, y_0) + d_m \int_0^t \mathbf{E}\big[g(x_{s\wedge\gamma_{m,n,k}}, y_{s\wedge\gamma_{m,n,k}})\big]\mathrm{d}s. \end{aligned} \tag{2.3}$$

Using (2.3) and Gronwall's lemma we obtain that for all $k \geq 1$,

$$\mathbf{E}\big[g(x_{t\wedge\gamma_{m,n,k}}, y_{t\wedge\gamma_{m,n,k}})\big] \leq g(x_0, y_0)\mathrm{e}^{d_m t}, \qquad t \geq 0.$$

Letting $k \to \infty$ we have

$$\mathbf{E}\big[g(x_{t\wedge\gamma_{m,n}}, y_{t\wedge\gamma_{m,n}})\big] \leq g(x_0, y_0)\mathrm{e}^{d_m t}, \qquad t \geq 0,$$

which implies that for each $m \geq 1$,

$$\mathbf{E}\Big[\lim_{n\to\infty} g(x_{t\wedge\gamma_{m,n}}, y_{t\wedge\gamma_{m,n}})\Big] \leq \liminf_{n\to\infty} \mathbf{E}\big[g(x_{t\wedge\gamma_{m,n}}, y_{t\wedge\gamma_{m,n}})\big] \leq g(x_0, y_0)\mathrm{e}^{d_m t} \tag{2.4}$$

by Fatou's lemma. From condition (i) and (2.2) it follows that $\mathbf{P}\{\tau_0 > t \wedge \sigma_m\} = 1$ for each $m \geq 1$ and $t > 0$. Letting $t \to \infty$ we get $\mathbf{P}\{\tau_0 \geq \sigma_m\} = 1$ for each $m \geq 1$. Thus, $\tau_0 \geq \lim_{m\to\infty} \sigma_m$ almost surely. Since these two processes are defined before the first time of hitting zero or explosion, then $\mathbf{P}\{\tau_0 = \infty \text{ or } \lim_{n\to\infty} \sigma_n = \infty\} = 1$. This concludes that $\tau_0 = \infty$ almost surely. □

**Proposition 2.2.** *Suppose that* $\sup_{t\geq 0}(x_t + y_t) < \infty$ *almost surely. We also assume that there exist a nonnegative function* $g \in C^2((0,\infty)\times(0,\infty))$ *and a sequence of nonnegative processes* $(d_m)_{m\geq 1}$ *satisfying the following conditions:*

(i) $0 < \sup_{x,y>0} g(x, y) < \infty$;
(ii) $\int_0^\infty d_m(t)\mathrm{d}t = \infty$ *almost surely for all* $m \geq 1$;
(iii) $L_t g(x_t, y_t) \geq d_m(t) g(x_t, y_t)$ *for all* $0 < t < \sigma_m$ *and large* $m \geq 1$.

*Then* $\mathbf{P}\{\tau_0 < \infty\} \geq g(x_0, y_0)/\sup_{x,y>0} g(x, y)$.

**Proof.** Let $D_m(t) := \int_0^t d_m(s)\mathrm{d}s$. Then for all $m \geq 1$,

$$D_m(t) \to \infty \text{ almost surely as } t \to \infty \tag{2.5}$$

by condition (ii). Let $([M^g, M^g]_t)_{t\geq 0}$ be the quadratic variation process of $(M_t^g)_{t\geq 0}$. Then the mapping $t \mapsto [M^g, M^g]_t$ is right continuous with left limits. It follows that

$$\sup_{s\in[0,t]} [M^g, M^g]_s < \infty \tag{2.6}$$

almost surely for all $t > 0$. For $n, m, k \geq 1$ define stopping times $\gamma_n$ and $\gamma_{m,n,k}$ by

$$\gamma_k := \inf\{t \geq 0 : [M^g, M^g]_t \geq k\}, \quad \gamma_{m,n,k} := \gamma_{m,n} \wedge \gamma_k.$$

Then

$$[M^g, M^g]_t \leq k \quad \text{for all } 0 \leq t < \gamma_k \quad \text{and} \quad k \geq 1 \tag{2.7}$$

and

$$\lim_{k\to\infty} \gamma_k = \infty \tag{2.8}$$

almost surely by (2.6). One can see that by the assumptions,

$$\lim_{m\to\infty} \mathbf{P}\{\sigma_m < \infty\} \leq \lim_{m\to\infty} \mathbf{P}\Big\{\sup_{t\geq 0}(x_t + y_t) \geq m\Big\} = 0. \tag{2.9}$$

Moreover, by [24, p. 73], $t \mapsto M^g_{t\wedge\gamma_{m,n,k}}$ is a martingale, where $M_t^g$ is defined in (2.1). It follows from integration by parts that

$$\begin{aligned}
&g(x_{t\wedge\gamma_{m,n,k}}, y_{t\wedge\gamma_{m,n,k}})\mathrm{e}^{-D_m(t)} \\
&= g(x_0, y_0) + \int_0^t g(x_{s\wedge\gamma_{m,n,k}}, y_{s\wedge\gamma_{m,n,k}})\mathrm{d}(\mathrm{e}^{-D_m(s)}) \\
&\quad + \int_0^t \mathrm{e}^{-D_m(s)}\mathrm{d}(g(x_{s\wedge\gamma_{m,n,k}}, y_{s\wedge\gamma_{m,n,k}})) \\
&= g(x_0, y_0) - \int_0^t g(x_{s\wedge\gamma_{m,n,k}}, y_{s\wedge\gamma_{m,n,k}}) d_m(s)\mathrm{e}^{-D_m(s)}\mathrm{d}s \\
&\quad + \int_0^t \mathrm{e}^{-D_m(s)} L_s g(x_s, y_s) 1_{\{s\leq\gamma_{m,n,k}\}}\mathrm{d}s + \int_0^t \mathrm{e}^{-D_m(s)}\mathrm{d}M^g_{s\wedge\gamma_{m,n,k}}.
\end{aligned} \tag{2.10}$$

By the Burkholder–Davis–Gundy inequality and (2.7), there is a constant $C > 0$ so that for all $T > 0$,

$$\begin{aligned}
&\mathbf{E}\Big[\sup_{0\le t\le T}\Big|\int_0^t \mathrm{e}^{-D_m(s)}\mathrm{d}M^g_{s\wedge\gamma_{m,n,k}}\Big|^2\Big]\\
&\quad\le C\mathbf{E}\Big[\Big|\int_0^T \mathrm{e}^{-2D_m(s)}\mathrm{d}[M^g, M^g]_{s\wedge\gamma_{m,n,k}}\Big|\Big]\\
&\quad\le C\mathbf{E}\Big[[M^g, M^g]_{T\wedge\gamma_{m,n,k}}\Big]\le Ck.
\end{aligned}$$

It then follows from [24, p. 38] that $t \mapsto \int_0^t \mathrm{e}^{-D_m(s)}\mathrm{d}M^g_{s\wedge\gamma_{m,n,k}}$ is a martingale. Taking expectations on both sides of (2.10) we get

$$\begin{aligned}
&\int_0^t \mathbf{E}\Big[d_m(s)\mathrm{e}^{-D_m(s)}g(x_{s\wedge\gamma_{m,n,k}}, y_{s\wedge\gamma_{m,n,k}})\Big]\mathrm{d}s + \mathbf{E}\Big[g(x_{t\wedge\gamma_{m,n,k}}, y_{t\wedge\gamma_{m,n,k}})\mathrm{e}^{-D_m(t)}\Big]\\
&\qquad= g(x_0, y_0) + \int_0^t \mathbf{E}\Big[\mathrm{e}^{-D_m(s)}L_s g(x_s, y_s)1_{\{s\le\gamma_{m,n,k}\}}\Big]\mathrm{d}s.
\end{aligned}$$

Letting $t\to\infty$ and using condition (i), (2.5) and the dominated convergence theorem we get

$$\begin{aligned}
&\int_0^\infty \mathbf{E}\Big[d_m(t)\mathrm{e}^{-D_m(t)}g(x_{t\wedge\gamma_{m,n,k}}, y_{t\wedge\gamma_{m,n,k}})\Big]\mathrm{d}t\\
&\quad= g(x_0, y_0) + \int_0^\infty \mathbf{E}\Big[\mathrm{e}^{-D_m(t)}L_t g(x_t, y_t)1_{\{t\le\gamma_{m,n,k}\}}\Big]\mathrm{d}t.
\end{aligned}$$

Using condition (iii) we have

$$\begin{aligned}
&\int_0^\infty \mathbf{E}\Big[d_m(t)\mathrm{e}^{-D_m(t)}g(x_{t\wedge\gamma_{m,n,k}}, y_{t\wedge\gamma_{m,n,k}})\Big]\mathrm{d}t\\
&\quad\ge g(x_0, y_0) + \int_0^\infty \mathbf{E}\Big[d_m(t)\mathrm{e}^{-D_m(t)}g(x_t, y_t)1_{\{t\le\gamma_{m,n,k}\}}\Big]\mathrm{d}t,
\end{aligned}$$

which implies

$$\begin{aligned}
g(x_0, y_0) &\le \mathbf{E}\Big[\int_0^\infty d_m(t)\mathrm{e}^{-D_m(t)}g(x_{\gamma_{m,n,k}}, y_{\gamma_{m,n,k}})1_{\{t>\gamma_{m,n,k}\}}\mathrm{d}t\Big]\\
&\le c_0\mathbf{E}\Big[\int_{\gamma_{m,n,k}}^\infty d_m(t)\mathrm{e}^{-D_m(t)}\mathrm{d}t\Big] = c_0\mathbf{E}\big[\mathrm{e}^{-D_m(\gamma_{m,n,k})}\big]
\end{aligned}$$

by condition (i) and (2.5) again, where $c_0 := \sup_{x,y>0} g(x, y)$. Letting $n, k\to\infty$ and using (2.2) and (2.8) we get

$$\begin{aligned}
g(x_0, y_0) &\le c_0\mathbf{E}\big[\mathrm{e}^{-D_m(\tau_0\wedge\sigma_m)}\big] = c_0\mathbf{E}\big[\mathrm{e}^{-D_m(\tau_0\wedge\sigma_m)}(1_{\{\sigma_m<\infty\}} + 1_{\{\sigma_m=\infty\}})\big]\\
&\le c_0\mathbf{P}\{\sigma_m < \infty\} + c_0\mathbf{E}\big[\mathrm{e}^{-D_m(\tau_0)}\big].
\end{aligned}$$

By (2.9), for each $\varepsilon\in(0,1)$, there is a large enough $m\ge 1$ so that

$$c_0\mathbf{P}\{\sigma_m < \infty\} \le \varepsilon g(x_0, y_0),$$

which means that

$$(1-\varepsilon)g(x_0, y_0) \le c_0\mathbf{E}\big[\mathrm{e}^{-D_m(\tau_0)}\big] \le c_0\mathbf{E}\big[\mathrm{e}^{-D_m(\tau_0)}1_{\{\tau_0=\infty\}} + 1_{\{\tau_0<\infty\}}\big] = c_0\mathbf{P}\{\tau_0 < \infty\},$$

where (2.5) is used in the last equation. Taking $\varepsilon\to 0$ one ends the proof. □

By an argument similar to that in the proof of Proposition 2.2, we obtain the next result.

**Corollary 2.3.** *Suppose that* $\sup_{t\geq 0}(x_t + y_t) < \infty$ *almost surely and* $g \in C^2((0,\infty)\times(0,\infty))$ *is a nonnegative function with* $0 < \sup_{x,y>0} g(x, y) < \infty$. *If there exist a constant* $\varepsilon > 0$ *and a nonnegative function* $h$ *on* $(0,\infty)$ *so that*

$$L_t g(x_t, y_t) \geq h(x_t) g(x_t, y_t), \qquad 0 < t < \sigma_\varepsilon$$

*and* $\int_0^\infty h(x_t \wedge \varepsilon)\mathrm{d}t = \infty$ *almost surely, then*

$$\mathbf{P}\{\tau_0 \wedge \sigma_\varepsilon < \infty\} \geq g(x_0, y_0)/\sup_{x,y>0} g(x, y).$$

**Proof.** We can prove the assertion with $d_m(t)$ and $\tau_0$ respectively replaced by $h(x_t \wedge \varepsilon)$ and $\tau_0 \wedge \sigma_\varepsilon$ in the proof of Proposition 2.2. We leave the details of the proof to the readers. □

Similar to Propositions 2.1 and 2.2, we can also obtain the associated assertions for the one-dimensional processes. Suppose that $x := (x_t)_{t\geq 0}$ is a non-negative Markov process and the operator $L_t$ is defined in the following: for each $g \in C^2((0,\infty))$ and $m, n \geq 1$, $t \mapsto M^g_{t\wedge\tilde{\gamma}_{m,n}}$ is a local martingale, where

$$M^g_t := g(x_t) - g(x_0) - \int_0^t L_s g(x_s)\mathrm{d}s$$

and $\tilde{\gamma}_{m,n} := \tau^x_{1/n} \wedge \sigma^x_m$.

**Corollary 2.4.** *Suppose that there are a non-negative function* $g \in C^2((0,\infty))$ *and constants* $d_m \geq 0$, $m \geq 1$ *satisfying* $\lim_{y\to 0} g(y) = \infty$ *and* $L_t g(y) \leq d_m g(y)$ *for all* $m \geq 1$, $y \in (0, m)$ *and* $t > 0$. *Then* $\mathbf{P}\{\tau^x_0 < \infty\} = 0$.

**Corollary 2.5.** *Suppose that* $\sup_{t\geq 0} x_t < \infty$ *almost surely, and that there exist a nonnegative function* $g \in C^2((0,\infty))$ *and a sequence of nonnegative processes* $(d_n)_{n\geq 1}$ *so that* $0 < \sup_{y>0} g(y) < \infty$, $\int_0^\infty d_n(t)\mathrm{d}t = \infty$ *almost surely and* $L_t g(x_t) \geq d_n(t) g(x_t)$ *for all* $0 < t < \sigma_n$ *and* $n \geq 1$. *Then* $\mathbf{P}\{\tau^x_0 < \infty\} \geq g(x_0)/\sup_{x>0} g(x)$.

## 3. Proofs of the main results

In this section we establish the proofs of Theorems 1.4–1.5 and 1.7–1.10. We first state some notations and assertions which will be used in the proofs. For $g \in C^2((0,\infty)\times(0,\infty))$ we define

$$\begin{aligned} K^1_z g(x, y) &:= g(x+z, y) - g(x, y) - z g'_x(x, y), \\ K^2_z g(x, y) &:= g(x, y+z) - g(x, y) - z g'_y(x, y) \end{aligned} \tag{3.1}$$

for $x, y, z > 0$ and

$$Lg(x, y) := L_1 g(x, y) + L_2 g(x, y) \tag{3.2}$$

with

$$L_1 g(x, y) := -a_1(x) g'_x(x, y) + \frac{1}{2} a_2(x) g''_{xx}(x, y) + a_3(x)\int_0^\infty K^1_z g(x, y)\mu(\mathrm{d}z) \tag{3.3}$$

and

$$L_2 g(x, y) := -[b_1(y) + \kappa(x)\theta(y)]g'_y(x, y) \\ +\frac{1}{2}b_2(y)g''_{yy}(x, y) + b_3(y)\int_0^\infty K_z^2 g(x, y)\nu(\mathrm{d}z), \tag{3.4}$$

where $g'_x, g''_{xx}$ and $g'_y, g''_{yy}$ denote the first and the second partial derivatives of $g$ with respect to $x$ and $y$. By (1.1) and Itô's formula, $L$ is the generator of $(X, Y)$ and independent of time $t$. By Taylor's formula, for any bounded continuous second derivative function $g$,

$$K_z g(x) = z^2 \int_0^1 g''(x + zv)(1 - v)\mathrm{d}v, \tag{3.5}$$

where

$$K_z g(x) := g(x + z) - g(x) - zg'(x), \qquad x, z > 0. \tag{3.6}$$

Since for all $x \in \mathbb{R}$, $\mathrm{e}^x - 1 \geq x$, then by (3.5), for all $x, y, z, \lambda > 0$,

$$\begin{aligned} &\mathrm{e}^{\lambda y^r}[\mathrm{e}^{-\lambda(y+z)^r} - \mathrm{e}^{-\lambda y^r} + \lambda r z y^{r-1}\mathrm{e}^{-\lambda y^r}] \\ &\quad = [\mathrm{e}^{\lambda y^r - \lambda(y+z)^r} - 1 + \lambda r z y^{r-1}] \\ &\quad \geq -\lambda[(y + z)^r - y^r - rzy^{r-1}] \\ &\quad = \lambda r(1 - r)z^2 \int_0^1 (y + zv)^{r-2}(1 - v)\mathrm{d}v \\ &\quad = \lambda r(1 - r)z^2 y^{r-2} \int_0^1 (1 + zy^{-1}v)^{r-2}(1 - v)\mathrm{d}v. \end{aligned} \tag{3.7}$$

Moreover, for $0 < r < 1$,

$$\begin{aligned} &\mathrm{e}^{\lambda y^r}[\mathrm{e}^{-\lambda(y+z)^r} - \mathrm{e}^{-\lambda y^r} + \lambda r z y^{r-1}\mathrm{e}^{-\lambda y^r}] \\ &\quad \geq \lambda r(1 - r)z^2 y^{r-2} \int_0^1 (1 + zy^{-1}v)^{-2}(1 - v)\mathrm{d}v. \end{aligned} \tag{3.8}$$

### *3.1. Preliminary results*

**Lemma 3.1.** *For any $u, v \geq 0$ and $\bar{p}, \bar{q} > 1$ with $1/\bar{p} + 1/\bar{q} = 1$, we have*

$$u + v \geq \bar{p}^{1/\bar{p}}\bar{q}^{1/\bar{q}}u^{1/\bar{p}}v^{1/\bar{q}}.$$

**Proof.** The above inequality follows from the Young inequality. □

Recall the definitions of $\tau_0^Z$ and $\tau_w^Z$ in (1.8) for the process $Z$ and constant $w > 0$.

**Lemma 3.2.**

(i) *For any $0 < w_1 < X_0$ and $0 < w_2 < Y_0$, we have $\tau_{w_1}^X < \infty$ and $\tau_{w_2}^Y < \infty$ almost surely. Moreover, $\lim_{t\to\infty} X_t = 0$ and $\lim_{t\to\infty} Y_t = 0$ almost surely.*
(ii) $\mathbf{P}\{\tau_0^X = \infty\} = 1$ *and* $\mathbf{P}\{\tau_0^Y = \infty\} = 1$ *if $\kappa(x) = 0$ for all $x > 0$.*

**Proof.** Observe that for each $w > 1$, $(1 + zu^{-1}v)^{-1-w} \leq (1 + zu^{-1}v)^{-2}$. Then

$$u^{-2}\int_0^\infty z^2\mu(\mathrm{d}z)\int_0^1 (1 + zu^{-1}v)^{-1-w}(1 - v)\mathrm{d}v \leq H_{1,0}(u),$$

$$u^{-2}\int_0^\infty z^2\nu(\mathrm{d}z)\int_0^1(1+zu^{-1}v)^{-1-w}(1-v)\mathrm{d}v\le H_{2,0}(u),$$

where $H_{1,0}$ and $H_{2,0}$ are defined in (1.4) and (1.5). It is obvious that condition (C2) is stronger than the assumption in Theorem 2.3 (i) of [20]. Thus, under conditions (C2) and (C3), applying Theorem 2.3 (i) and Proposition 2.6 of [20] we prove the assertions. □

The next result gives an estimate on the maximums of processes $X$ and $Y$.

**Lemma 3.3.** *Given $X_0, Y_0 > 0$, for any $\delta \in (0, 1/2)$ and $\varepsilon > 0$, there exists a constant $C > 0$ that does not depend on $\varepsilon, X_0, Y_0$ so that*

$$\mathbf{P}\Big\{\sup_{t\ge 0} X_t\ge\varepsilon\Big\}\le C\varepsilon^{-\delta}X_0^\delta,\quad \mathbf{P}\Big\{\sup_{t\ge 0} Y_t\ge\varepsilon\Big\}\le C\varepsilon^{-\delta}Y_0^\delta.$$

**Proof.** Observe that

$$(X_s+z)^{2\delta}-X_s^{2\delta}-2\delta X_s^{2\delta-1}z=X_s^{2\delta}\big[(1+zX_s^{-1})^{2\delta}-1+(-2\delta)zX_s^{-1}\big]$$

and then

$$\int_0^\infty\big[(X_s+z)^{2\delta}-X_s^{2\delta}-2\delta X_s^{2\delta-1}z\big]\mu(\mathrm{d}z)=-2\delta(1-2\delta)X_s^{2\delta}H_{1,-2\delta}(X_s),$$

where the function $H_{1,-2\delta}$ is defined in (1.2). Then by (1.1) and Itô's formula, we have

$$\begin{aligned}X_t^{2\delta} &= X_0^{2\delta}-2\delta\int_0^t a_1(X_s)X_s^{2\delta-1}\mathrm{d}s-\delta(1-2\delta)\int_0^t a_2(X_s)X_s^{2\delta-2}\mathrm{d}s\\ &\quad -2\delta(1-2\delta)\int_0^t a_3(X_s)X_s^{2\delta}H_{1,-2\delta}(X_s)\mathrm{d}s+2\delta\int_0^t a_2(X_s)^{1/2}X_s^{2\delta-1}\mathrm{d}B_s\\ &\quad +\int_0^t\int_0^\infty\int_0^{a_3(X_{s-})}[(X_{s-}+z)^{2\delta}-X_{s-}^{2\delta}]\tilde{M}(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u)\\ &=: X_0^{2\delta}-\sum_{i=1}^5 A_i(t,2\delta).\end{aligned}\tag{3.9}$$

Since $0<\delta<1/2$ and $a_1, a_2, a_3$ are nonnegative by condition (C1), then $A_i(t, 2\delta)\ge 0$ for all $t\ge 0$ and $i = 1, 2, 3$. With $2\delta$ replaced by $\delta$ in (3.9) it follows that

$$X_t^\delta\le X_0^\delta+|A_4(t,\delta)|+|A_5(t,\delta)|.\tag{3.10}$$

For all $n\ge 1$ let $\tilde{\tau}_n := \tau_{1/n}^X\wedge\sigma_n^X$. Then

$$\mathbf{E}[A_4(t\wedge\tilde{\tau}_n,2\delta)]=\mathbf{E}[A_5(t\wedge\tilde{\tau}_n,2\delta)]=0.$$

It follows from (3.9) that $\mathbf{E}[A_2(t\wedge\tilde{\tau}_n, 2\delta)]\le X_0^{2\delta}$. We then apply Fatou's lemma to get

$$\mathbf{E}\Big[\sup_{t\ge 0}A_2(t,2\delta)\Big]=\mathbf{E}\Big[\lim_{t,n\to\infty}A_2(t\wedge\tilde{\tau}_n,2\delta)\Big]\le\liminf_{t,n\to\infty}\mathbf{E}[A_2(t\wedge\tilde{\tau}_n,2\delta)]\le X_0^{2\delta}.\tag{3.11}$$

Similarly, we can also get

$$\mathbf{E}\Big[\sup_{t\ge 0}A_3(t,2\delta)\Big]\le X_0^{2\delta}.\tag{3.12}$$

By the Burkholder–Davis–Gundy inequality, the Hölder inequality and the estimate (3.11), there is a constant $C_1 > 0$ so that

$$\begin{aligned}\mathbf{E}\Big[\sup_{t\geq 0}|A_4(t,\delta)|\Big] &\leq \delta C_1 \mathbf{E}\Big[\Big|\int_0^\infty a_2(X_s)X_s^{2\delta-2}\mathrm{d}s\Big|^{1/2}\Big] \\ &\leq \delta C_1 \Big|\mathbf{E}\Big[\int_0^\infty a_2(X_s)X_s^{2\delta-2}\mathrm{d}s\Big]\Big|^{1/2} \\ &\leq \delta C_1[\delta(1-2\delta)]^{-1/2}\Big|\mathbf{E}\Big[\sup_{t\geq 0}A_2(t,2\delta)\Big]\Big|^{1/2} \\ &\leq \delta C_1[\delta(1-2\delta)]^{-1/2}X_0^\delta. \end{aligned} \tag{3.13}$$

For fixed $z, x \geq 0$ and $u \geq 0$ let $h(u) = (1 + zx^{-1}u)^{2\delta-2}$. Then $h$ is decreasing and $(h(u) - h(1/2))(1/2 - u) \geq 0$ for all $u \geq 0$. It follows that

$$\int_0^1 (h(u) - h(1/2))(1/2 - u)\mathrm{d}u \geq 0.$$

Since $\int_0^1 h(1/2)(1/2 - u)\mathrm{d}u = 0$, then $\int_0^1 h(u)(1/2 - u)\mathrm{d}u \geq 0$. Moreover,

$$\int_0^1 h(u)(1-u)\mathrm{d}u = \frac{1}{2}\int_0^1 h(u)\mathrm{d}u + \int_0^1 h(u)(1/2-u)\mathrm{d}u \geq \frac{1}{2}\int_0^1 h(u)\mathrm{d}u.$$

Combining this with Taylor's formula and Hölder's inequality we further have

$$\begin{aligned} &[(x+z)^\delta - x^\delta]^2 = \delta^2\Big|z\int_0^1 (x+zu)^{\delta-1}\mathrm{d}u\Big|^2 \\ &= \delta^2 x^{2\delta-2}z^2\Big|\int_0^1 (1+zx^{-1}u)^{\delta-1}\mathrm{d}u\Big|^2 \\ &\leq \delta^2 x^{2\delta-2}z^2\int_0^1 (1+zx^{-1}u)^{2\delta-2}\mathrm{d}u \\ &\leq 2\delta^2 x^{2\delta-2}z^2\int_0^1 (1+zx^{-1}u)^{2\delta-2}(1-u)\mathrm{d}u. \end{aligned}$$

Then by the Burkholder–Davis–Gundy inequality and the Hölder inequality, there are constants $C_2 > 0$ and $C_3 = C_3(\delta) > 0$ so that

$$\begin{aligned}\mathbf{E}\Big[\sup_{t\geq 0}|A_5(t,\delta)|\Big] &\leq \delta C_2 \mathbf{E}\Big[\Big|\int_0^\infty a_3(X_s)\mathrm{d}s\int_0^\infty [(X_s+z)^\delta - X_s^\delta]^2\mu(\mathrm{d}z)\Big|^{1/2}\Big] \\ &\leq \delta C_2\Big|\mathbf{E}\Big[\int_0^\infty a_3(X_s)\mathrm{d}s\int_0^\infty [(X_s+z)^\delta - X_s^\delta]^2\mu(\mathrm{d}z)\Big]\Big|^{1/2} \\ &\leq 2^{1/2}\delta^2 C_2\Big|\mathbf{E}\Big[\int_0^\infty a_3(X_s)X_s^{2\delta}H_{1,-2\delta}(X_s)\mathrm{d}s\Big]\Big|^{1/2} \\ &= C_3\Big|\mathbf{E}\Big[\sup_{t\geq 0}A_3(t,2\delta)\Big]\Big|^{1/2} \leq C_3 X_0^\delta, \end{aligned} \tag{3.14}$$

where (3.12) is used in the last inequality. Combining (3.14) with (3.10) and (3.13) we have

$$\mathbf{E}\Big[\sup_{t\geq 0}X_t^\delta\Big] \leq C_4 X_0^\delta$$

for some constant $C_4 > 0$ independent of $X_0$. Then by the Markov inequality,

$$\mathbf{P}\Big\{\sup_{t\ge 0} X_t > \varepsilon\Big\} \le \varepsilon^{-\delta}\mathbf{E}\Big[\sup_{t\ge 0} X_t^{\delta}\Big] \le C_4\varepsilon^{-\delta}X_0^{\delta}.$$

By the same argument we can show that

$$\mathbf{P}\Big\{\sup_{t\ge 0} Y_t > \varepsilon\Big\} \le C_5\varepsilon^{-\delta}Y_0^{\delta}$$

for some constant $C_5 > 0$. This concludes the proof. □

### 3.2. *Proof of Theorem 1.4*

**Proof of Theorem 1.4.** We apply Proposition 2.1 to prove Theorem 1.4. The key is to construct a function $g$ that satisfies the conditions (i) and (ii) in Proposition 2.1. For $\rho > 0$ we choose the function $g$ as

$$g(x, y) = x^{-\rho} + y^{-\rho} + 1, \qquad x, y > 0.$$

Then for all $x, y > 0$,

$$\begin{aligned} g'_x(x, y) &= -\rho x^{-\rho-1}, \quad & g''_{xx}(x, y) &= \rho(\rho+1)x^{-\rho-2},\\ g'_y(x, y) &= -\rho y^{-\rho-1}, \quad & g''_{yy}(x, y) &= \rho(\rho+1)y^{-\rho-2}. \end{aligned}$$

It thus follows that for $x, y > 0$,

$$-g'_x(x, y) \le \rho x^{-1}g(x, y), \quad g''_{xx}(x, y) \le \rho(\rho+1)x^{-2}g(x, y) \tag{3.15}$$

and

$$-g'_y(x, y) \le \rho y^{-1}g(x, y), \quad g''_{yy}(x, y) \le \rho(\rho+1)y^{-2}g(x, y). \tag{3.16}$$

Moreover, by (3.1), for $x, y, z > 0$,

$$\begin{aligned} K_z^1 g(x, y) &= z^2\int_0^1 g''_{xx}(x + zv, y)(1 - v)\mathrm{d}v\\ &= \rho(\rho+1)z^2\int_0^1 (x + zv)^{-\rho-2}(1 - v)\mathrm{d}v\\ &\le \rho(\rho+1)g(x, y)x^{-2}z^2\int_0^1 (1 + zx^{-1}v)^{-2}(1 - v)\mathrm{d}v. \end{aligned}$$

Thus

$$\int_0^\infty K_z^1 g(x, y)\mu(\mathrm{d}z) \le \rho(\rho+1)g(x, y)H_{1,0}(x), \quad x, y > 0, \tag{3.17}$$

where the function $H_{1,0}$ is defined in (1.4). Similarly, we can obtain

$$\int_0^\infty K_z^2 g(x, y)\nu(\mathrm{d}z) \le \rho(\rho+1)g(x, y)H_{2,0}(y), \quad x, y > 0, \tag{3.18}$$

where the function $H_{2,0}$ is defined in (1.5).

Recalling (3.2)–(3.4) and combining (3.15)–(3.18), we get

$$L_1 g(x, y) \le \rho(\rho+1)G_{1,0}(x)g(x, y)$$

and

$$L_2 g(x,y) \le \rho(\rho+1)G_{2,0}(y)g(x,y) + \rho\kappa(x)\theta(y)y^{-1}g(x,y)$$

for $x, y > 0$, where $G_{1,0}$ and $G_{2,0}$ are defined in (1.6) and (1.7), respectively. Then under conditions (C1) and (C2) and the assumption of the theorem, $g(x,y)^{-1}Lg(x,y)$ is bounded for $x, y$ in any bounded interval. Therefore, for each $n \ge 1$, there is a constant $d_n > 0$ so that $Lg(x,y) \le d_n g(x,y)$ for all $x, y \in (0,n)$, which implies condition (ii) in Proposition 2.1. By the definition of $g$, condition (i) in Proposition 2.1 are obvious. Since $\tau_0^X = \infty$, $\mathbf{P}$-a.s. by Lemma 3.2(ii), we have $\tau_0 = \tau_0^Y$, $\mathbf{P}$-a.s., where $\tau_0 = \tau_0^X \wedge \tau_0^Y$ by Remark 1.2. It follows from the assertion in Proposition 2.1 that $\mathbf{P}\{\tau_0^Y = \infty\} = 1$. $\square$

### 3.3. Proof of Theorem 1.5

**Proof of Theorem 1.5.** We want to apply Proposition 2.2 where the key is to construct a function $g$ satisfying the conditions (i)–(iii) in Proposition 2.2. We assume that the constant $c^*$ in the assumption satisfies $0 < c^* < 1$. By condition (C3), there is a small enough constant $c_2 \in (0, c^*)$ so that $\int_{c_2}^{c^*} \mu(\mathrm{d}z) > 0$. Let $0 < c_1 < c_2 < c_3 < c^*$. Choose a constant $c_0 > 0$ so that

$$\inf_{c_1 \le u \le c^*} \kappa(u) \ge c_0, \quad \inf_{0<u\le c^*} \theta(u)u^{-\theta} \ge c_0 \tag{3.19}$$

and

$$\inf_{0<u\le c^*} \left[a_2(u)u^{-2-\delta} + a_3(u)u^{-\delta-1}\right] \ge c_0, \tag{3.20}$$

where $\delta > 1$ is the constant appearing in the assumption. The proof is given in the following three steps.

**Step 1.** In this step we construct the function $g$ and summarize some of its properties. Let $g_0 \in C^2((0,c^*))$ satisfy $g_0(x) = x^{-\delta}$ for $x \in (0,c_2)$ and $g_0(x) = (x-c^*)^{-2}$ for $x \in (c_3, c^*)$. We choose function $g_0$ so that $g_0, g_0'$ and $g_0''$ are all bounded in $[c_2, c_3]$. For $\lambda_1, \lambda_2 > 1$, $\bar{c} := \pi/(2c^*)$ and $0 < r < 1-\theta$, define a nonnegative function $g$ by

$$g(x,y) := \exp\{-\lambda_1 g_0(x) - \lambda_2(\tan \bar{c}y)^r\}1_{\{x,y<c^*\}}, \qquad x, y > 0,$$

where we only need the properties of a tan function such that it is equivalent to $x$ near zero and is infinite at $\pi/2$. Then $g \in C^2((0,\infty)\times(0,\infty))$, and for $0 < x, y < c^*$,

$$g_x'(x,y) = -\lambda_1 g_0'(x)g(x,y), \quad g_y'(x,y) = -\lambda_2 \bar{c} r(\tan \bar{c}y)^{r-1}(\cos \bar{c}y)^{-2}g(x,y), \tag{3.21}$$

and

$$g_{xx}''(x,y) = \lambda_1\left[\lambda_1|g_0'(x)|^2 - g_0''(x)\right]g(x,y), \tag{3.22}$$

$$\begin{aligned} g_{yy}''(x,y) &= \lambda_2 r\bar{c}^2 g(x,y)(\sin \bar{c}y)^{2r-2}(\cos \bar{c}y)^{-2-2r}[\lambda_2 r \\ &\qquad +(1-r)(\sin \bar{c}y)^{-r}(\cos \bar{c}y)^r - 2(\sin \bar{c}y)^{2-r}(\cos \bar{c}y)^r] \\ &\ge \lambda_2 r\bar{c}^2 g(x,y)(\sin \bar{c}y)^{2r-2}(\cos \bar{c}y)^{-2-2r}(\lambda_2 r - 2) \\ &\ge 2^{-1}(\lambda_2 r\bar{c})^2 g(x,y)(\sin \bar{c}y)^{2r-2} \end{aligned} \tag{3.23}$$

as $\lambda_2 \ge 4r^{-1}$. Observe that the constant $\delta > 1$ by the assumption of the theorem. Taking $\lambda_1$ large enough so that $\lambda_1\delta - (\delta+1)c_2^\delta \ge \lambda_1$ and $2\lambda_1 \ge 3|c^*|^2$ in the following, by (3.22) we get

$$\begin{aligned} g''_{xx}(x,y)/g(x,y) &= \lambda_1^2\delta^2 x^{-2\delta-2} - \lambda_1\delta(\delta+1)x^{-\delta-2} \\ &= \lambda_1\delta x^{-2\delta-2}[\lambda_1\delta - (\delta+1)x^{\delta}] \\ &\geq \lambda_1\delta x^{-2\delta-2}[\lambda_1\delta - (\delta+1)c_2^{\delta}] \\ &\geq \lambda_1^2\delta x^{-2\delta-2}, \qquad 0<x<c_2,\ y>0 \end{aligned} \tag{3.24}$$

and

$$g''_{xx}(x,y)/g(x,y) = 2\lambda_1(x-c^*)^{-6}[2\lambda_1 - 3(x-c^*)^2] > 0, \quad c_3<x<c^*,\ y>0. \tag{3.25}$$

In addition, since $g_0$, $g_0'$ and $g_0''$ are bounded on $[c_1, c_3]$, then

$$C_0 := \sup_{x\geq c_1, y>0}\Big[g(x,y) + |g'_x(x,y)| + |g''_{xx}(x,y)|\Big] < \infty. \tag{3.26}$$

**Step 2.** In this step, we estimate $L_1 g(x,y)$ which is defined in (3.3). Recall (3.1). Observe that $g(u,y)\geq g(x,y)$ for all $0<x\leq u<c_2$ and $y>0$. It follows from (3.1), (3.5) and (3.24) that for $x<c_1$, $z<c_2-c_1$ and $0<y<c^*$ we have

$$\begin{aligned} g(x,y)^{-1}K_z^1 g(x,y) &= z^2 g(x,y)^{-1}\int_0^1 g''_{xx}(x+zu,y)(1-u)\mathrm{d}u \\ &\geq \lambda_1^2 z^2 g(x,y)^{-1}\int_0^1 (x+zu)^{-2\delta-2} g(x+zu,y)(1-u)\mathrm{d}u \\ &\geq \lambda_1^2 z^2 \int_0^1 (x+zu)^{-2\delta-2}(1-u)\mathrm{d}u \\ &\geq \lambda_1^2 x^{-2\delta-2} z^2 \int_0^x (1+c_2-c_1)^{-2\delta-2}(1-u)\mathrm{d}u \geq \lambda_1^2 C_1 x^{-2\delta-1} z^2 \end{aligned}$$

for some constant $C_1>0$ independent of $\lambda_1$ and $\lambda_2$, which gives

$$\int_0^{c_2-c_1} K_z^1 g(x,y)\mu(\mathrm{d}z) \geq \lambda_1^2 C_1 \int_0^{c_2-c_1} z^2\mu(\mathrm{d}z) x^{-2\delta-1} g(x,y), \quad x\leq c_1,\ y>0. \tag{3.27}$$

Since $\lambda_1, \delta > 1$, then by (3.1) and (3.21),

$$\begin{aligned} K_z^1 g(x,y) &\geq -g(x,y) - z g'_x(x,y) = -g(x,y) - \lambda_1\delta z x^{-\delta-1} g(x,y) \\ &\geq -\lambda_1\delta x^{-\delta-1} g(x,y)(1+z), \qquad x\leq c_1,\ y>0, \end{aligned}$$

which implies that

$$\int_{c_2-c_1}^{\infty} K_z^1 g(x,y)\mu(\mathrm{d}z) \geq -\lambda_1\delta x^{-\delta-1} g(x,y)\int_{c_2-c_1}^{\infty}(1+z)\mu(\mathrm{d}z), \quad x\leq c_1,\ y>0. \tag{3.28}$$

Combining (3.27) and (3.28) we get

$$g(x,y)^{-1}\int_0^{\infty} K_z^1 g(x,y)\mu(\mathrm{d}z) \geq \lambda_1^2 C_1 \int_0^{c_2-c_1} z^2\mu(\mathrm{d}z) x^{-2\delta-1} - \lambda_1\delta x^{-\delta-1}\int_{c_2-c_1}^{\infty}(1+z)\mu(\mathrm{d}z)$$

for $0<x<c_1$ and $0<y<c^*$. Therefore, by (3.21) and (3.24),

$$\begin{aligned} & g(x,y)^{-1} L_1 g(x,y) \\ &\geq \lambda_1 x^{-\delta}\Big[-\delta a_1(x) x^{-1} + \lambda_1 2^{-1}\delta a_2(x) x^{-2-\delta} \\ &\quad + \Big(\lambda_1 C_1\int_0^{c_2-c_1} z^2\mu(\mathrm{d}z) - \delta x^{\delta}\int_{c_2-c_1}^{\infty}(1+z)\mu(\mathrm{d}z)\Big) a_3(x) x^{-\delta-1}\Big] \end{aligned} \tag{3.29}$$

for all $0 < x < c_1, 0 < y < c^*$. Under condition (C2), $C_2 := \sup_{0<x<c^*} a_1(x)x^{-1} < \infty$. Combining this with (3.29) and (3.20), for all $0 < x < c_1, 0 < y < c^*$ and large enough $\lambda_1$ with

$$2^{-1}\lambda_1 C_1 \int_0^{c_2-c_1} z^2\mu(\mathrm{d}z) > \delta c_1^\delta \int_{c_2-c_1}^\infty (1+z)\mu(\mathrm{d}z),$$

we get

$$\begin{aligned}
&g(x, y)^{-1}L_1 g(x, y)\\
&\geq \lambda_1 c_1^{-\delta}\Big[-C_2\delta + \lambda_1 2^{-1}\delta a_2(x)x^{-2-\delta}\\
&\qquad + \Big(\lambda_1 C_1 \int_0^{c_2-c_1} z^2\mu(\mathrm{d}z) - \delta c_1^\delta \int_{c_2-c_1}^\infty (1+z)\mu(\mathrm{d}z)\Big)a_3(x)x^{-\delta-1}\Big]\\
&\geq \lambda_1 c_1^{-\delta}\Big[-C_2\delta + 2^{-1}\lambda_1 a_2(x)x^{-2-\delta} + 2^{-1}\lambda_1 C_1 \int_0^{c_2-c_1} z^2\mu(\mathrm{d}z)a_3(x)x^{-\delta-1}\Big]\\
&\geq \lambda_1 c_1^{-\delta}\Big[-C_2\delta + 2^{-1}\lambda_1\Big[1\wedge\Big(C_1\int_0^{c_2-c_1} z^2\mu(\mathrm{d}z)\Big)\Big]\Big[a_2(x)x^{-2-\delta} + a_3(x)x^{-\delta-1}\Big]\Big]\\
&\geq \lambda_1 c_1^{-\delta}\Big[-C_2\delta + 2^{-1}\lambda_1 c_0\Big[1\wedge\Big(C_1\int_0^{c_2-c_1} z^2\mu(\mathrm{d}z)\Big)\Big]\Big].
\end{aligned} \tag{3.30}$$

Observe that the term in the bracket of the above inequality is positive for large enough $\lambda_1$. Thus for all large enough $\lambda_1 > 0$ there is a constant $d_1 := d_1(\lambda_1) > 0$ so that

$$g(x, y)^{-1}L_1 g(x, y) \geq d_1, \qquad 0 < x < c_1,\ 0 < y < c^*. \tag{3.31}$$

Since $g(x, y) = 0$ for all $x \geq c^*$ or $y \geq c^*$, then $L_1 g(x, y) = 0$ for all $x \geq c^*$ or $y \geq c^*$. By (3.21) and (3.25), for large enough $\lambda_1$,

$$\begin{aligned}
&-g'_x(x, y) = 2\lambda_1(c^*-x)^{-3}g(x, y),\ \ g''_{xx}(x, y) \geq 2\lambda_1^2(x-c^*)^{-6}g(x, y),\\
&c_3 < x < c^*,\ \ y > 0.
\end{aligned}$$

Thus, for large enough $\lambda_1 > 0$ we have $-g'_x(x, y) \geq 0$, $g''_{xx}(x, y) \geq 0$, and

$$K_z^1 g(x, y) = z^2 \int_0^1 g''_{xx}(x+zu, y)(1-u)\mathrm{d}u \geq 0$$

for all $x > c_3$ and $y > 0$. Now by the definition of $L_1 g(x, y)$ in (3.3),

$$L_1 g(x, y) \geq 0, \qquad x \geq c_3,\ y > 0 \tag{3.32}$$

for large enough $\lambda_1 > 0$. By (3.26), for each $c_1 \leq x \leq c_3$, $0 < y < c^*$ and $z > 0$,

$$|K_z^1 g(x, y)| = z^2\Big|\int_0^1 g''_{xx}(x+zu, y)(1-u)\mathrm{d}u\Big| \leq C_0 z^2$$

and

$$K_z^1 g(x, y) \geq -g(x, y) - zg'(x, y) \geq -C_0(1+z).$$

Then by (3.26) and the definition of $L_1 g(x, y)$ in (3.3) again, for $c_1 \leq x \leq c_3$ and $0 < y \leq c^*$,

$$L_1 g(x, y) \geq -C_0\Big[a_1(x) + 2^{-1}a_2(x) + a_3(x)\int_0^1 z^2\mu(\mathrm{d}z) + a_3(x)\int_1^\infty (1+z)\mu(\mathrm{d}z)\Big]. \tag{3.33}$$

Since $\inf_{c_1 \le x \le c_3,\, 0<y<c^*} g(x, y) > 0$, then by (3.33) and condition (C1), it is elementary to see that there is a constant $d_2 := d_2(\lambda_1) > 0$ so that

$$g(x, y)^{-1} L_1 g(x, y) \ge -d_2, \qquad c_1 \le x \le c_3,\ 0 < y < c^*. \tag{3.34}$$

Combining the above inequality with (3.31) and (3.32) we obtain

$$\begin{aligned} L_1 g(x, y) &\ge -d_2 g(x, y), \qquad x \ge c_1,\ 0 < y < c^*, \\ L_1 g(x, y) &\ge d_1 g(x, y), \qquad 0 < x < c_1,\ 0 < y < c^*. \end{aligned} \tag{3.35}$$

**Step 3.** In this step we first estimate $L_2 g(x, y)$ defined in (3.4) and then finish the proof. By (3.21) and (3.23),

$$-g'_y(x, y) \ge 0,\ g''_{yy}(x, y) \ge 0, \quad 0 < x \le c^*,\ 0 < y < c^*,\ \lambda_2 \ge 4r^{-1}. \tag{3.36}$$

Moreover, by (3.1) and (3.5),

$$\begin{aligned} &K_z^2 g(x, y) = z^2 \int_0^1 g''_{yy}(x, y + zv)(1 - v)\mathrm{d}v \ge 0, \\ &0 < x \le c^*,\ 0 < y < c^*,\ \lambda_2 \ge 4r^{-1}. \end{aligned} \tag{3.37}$$

Therefore, by the assumption of $0 < r < 1 - \theta$, (3.19) and (3.21) again,

$$\begin{aligned} L_2 g(x, y) &\ge -\kappa(x)\theta(y) g'_y(x, y) = \lambda_2 r \bar{c} \kappa(x)\theta(y)(\sin \bar{c} y)^{r-1}(\cos \bar{c} y)^{-1-r} g(x, y) \\ &\ge \lambda_2 r \bar{c} \kappa(x)\theta(y)(\bar{c} y)^{r-1} g(x, y) \ge \lambda_2 r \bar{c}^r c_0^2 y^{\theta-1+r} g(x, y) \ge 2d_2 g(x, y) \end{aligned} \tag{3.38}$$

for all $c_1 \le x \le c^*$, $0 < y < c^*$ and for $\lambda_2$ large enough, where the constant $d_2 > 0$ is determined in (3.34). Since $g(x, y) = 0$ for $x \ge c^*$ or $y \ge c^*$, then $L_2 g(x, y) = 0$ for $x \ge c^*$ or $y \ge c^*$. It follows from (3.38) that

$$L_2 g(x, y) \ge 2d_2 g(x, y), \quad x \ge c_1,\ y > 0 \tag{3.39}$$

and $L_2 g(x, y) \ge 0$ for all $x, y > 0$ by (3.36) and (3.37). Recalling (3.2). Combining (3.39) with (3.35) we get

$$\begin{aligned} Lg(x, y) &= L_1 g(x, y) + L_2 g(x, y) \ge d_1 g(x, y), \quad 0 < x < c_1,\ y > 0, \\ Lg(x, y) &\ge [2d_2 - d_2] g(x, y) = d_2 g(x, y), \qquad x \ge c_1,\ y > 0, \end{aligned}$$

which verifies condition (iii) of Proposition 2.2.

Therefore, by Proposition 2.2, $\mathbf{P}\{\tau_0^X \wedge \tau_0^Y < \infty\} \ge g(x_0, y_0)/[\sup_{x,y>0} g(x, y)]$ for large enough $\lambda_1, \lambda_2 > 0$ and $X_0, Y_0 \in (0, c^*)$. Since $\tau_0^X = \infty$ almost surely by Lemma 3.2(ii), we have $\mathbf{P}\{\tau_0^Y < \infty\} > 0$ for $0 < X_0, Y_0 < c^*$. For general initial values $X_0 > c^*$ or $Y_0 > c^*$, let $\tau := \tau_{c^*}^{X+Y}$. By Lemma 3.2 we have $\tau < \infty$ almost surely and then by the Markov property,

$$\mathbf{P}\{\tau_0^Y < \infty\} = \mathbf{P}\{\tau_0^Y < \infty | (X_\tau, Y_\tau)\} > 0,$$

which completes the proof. □

### 3.4. Proof of Theorem 1.7

**Lemma 3.4.** *Suppose that Condition 1.6(ib)–(ic) hold. Let $\tilde{g}$ be a nonnegative process satisfying $\int_0^\infty \tilde{g}(s)^\delta \mathrm{d}s = \infty$ almost surely for some constant $\delta$ with $q/(q + 1 - \theta) < \delta \le 1$. Let*

$(u_t)_{t\geq 0}$ *be the non-negative solution to*

$$u_t = u_0 - \int_0^t [b_1(u_s) + \theta(u_s)\tilde{g}(s)]\mathrm{d}s + \int_0^t b_2(u_s)^{1/2}\mathrm{d}W_s + \int_0^t\int_0^\infty\int_0^{b_3(u_{s-})} z\tilde{N}(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u).$$

*Then for each $u_0 > 0$, we have $\mathbf{P}\{\tau_0^u < \infty\} = 1$.*

**Proof.** To establish the proof we apply Corollary 2.5. The key is to construct the function $g$ and verify the conditions in Corollary 2.5. For $r \in (0, 1-\theta)$ and $v, \lambda > 0$ let $g(v) = \mathrm{e}^{-\lambda v^r}$. Then

$$g'(v) = -r\lambda v^{r-1}g(v), \;\; g''(v) = r\lambda[r\lambda v^r + (1-r)]v^{r-2}g(v) \geq r(1-r)\lambda v^{r-2}g(v). \tag{3.40}$$

By Itô's formula we can see that the operator $L_t$ is given by

$$L_t g(v) := -[b_1(v) + \theta(v)\tilde{g}(t)]g'(v) + 2^{-1}b_2(v)g''(v) + b_3(v)\int_0^\infty K_z g(v)\nu(\mathrm{d}z),$$

where $K_z g(v)$ is given in (3.6). In the following we find an estimation of $L_t g(v)$. It follows from (3.8) that

$$\int_0^\infty K_z g(v)\nu(\mathrm{d}z) \geq \lambda r(1-r)v^r H_{2,0}(v)g(v), \tag{3.41}$$

where the function $H_{2,0}$ is defined in (1.5). Therefore, by (3.40) and (3.41), for all $n \geq 1$,

$$\begin{aligned} L_t g(v) &\geq \lambda r g(v)[b_1(v)v^{r-1} + (1-r)2^{-1}b_2(v)v^{r-2} + (1-r)b_3(v)v^r H_{2,0}(v)] \\ &\geq \lambda r(1-r)g(v)v^r G_{2,0}(v) \geq \lambda r(1-r)|c^*|^r d_n g(v), \qquad c^* \leq v < n, \end{aligned} \tag{3.42}$$

where the function $G_{2,0}$ is defined in (1.7) and $d_n := \inf_{c^*\leq v<n} G_{2,0}(v) > 0$. Under Condition 1.6(ib)–(ic), we have

$$\begin{aligned} L_t g(v) &\geq \lambda r v^r g(v)\big[b_1(v)v^{-1} + (1-r)2^{-1}b_2(v)v^{-2} + (1-r)b_3(v)H_{2,0}(v) + c_\theta\tilde{g}(t)v^{\theta-1}\big] \\ &\geq \lambda r(1-r)v^r g(v)\big[G_{2,0}(v) + c_\theta\tilde{g}(t)v^{\theta-1}\big] \\ &\geq \lambda r(1-r)g(v)\big[bv^{q+r} + c_\theta\tilde{g}(t)v^{\theta-1+r}\big], \qquad 0 < v \leq c^*. \end{aligned}$$

Then

$$L_t g(v) \geq \lambda r(1-r)c_\theta g(v)\tilde{g}(t)|c^*|^{\theta-1+r}, \qquad 0 < v \leq c^*, \tag{3.43}$$

and by Lemma 3.1, there are constants $C_1 = C_1(r) > 0$ and $C_2 = C_2(r) > 0$ so that

$$\begin{aligned} g(v)^{-1}L_t g(v) &\geq C_1\lambda v^{(1-1/\bar{q})(r+q)+(\theta-1+r)/\bar{q}}\tilde{g}(t)^{1/\bar{q}} \\ &= C_1\lambda\tilde{g}(t)^{1/\bar{q}}v^{r+q-(q+1-\theta)/\bar{q}} \geq C_2\lambda\tilde{g}(t)^{1/\bar{q}}, \quad 0 < v \leq c^* \end{aligned} \tag{3.44}$$

for $\bar{q} > 1$ and $r + q - (q+1-\theta)/\bar{q} \leq 0$, which is equivalent to

$$\frac{1}{\bar{q}} \geq \frac{r+q}{q+1-\theta}.$$

It holds as long as $r$ is small enough and

$$\frac{1}{\bar{q}} > \frac{q}{q+1-\theta}.$$

Combining Corollary 2.5 and (3.42)–(3.44) one gets $\mathbf{P}\{\tau_0^u < \infty\} \geq \mathrm{e}^{-\lambda u_0^r}$ if either

$$\int_0^\infty \tilde{g}(s)\mathrm{d}s = \infty, \qquad \text{or} \int_0^\infty \tilde{g}(s)^{1/\bar{q}}\mathrm{d}s = \infty \quad \text{for } 1 > \frac{1}{\bar{q}} > \frac{r+q}{q+1-\theta}.$$

Taking $\bar{q} = 1/\delta$ and letting $\lambda \to 0$ we get $\mathbf{P}\{\tau_0^u < \infty\} = 1$ under the above conditions. This finishes the proof. $\square$

**Lemma 3.5.** *Suppose that Condition 1.6(ia) holds. Then for $0 < \bar{p} < 1$ and $\kappa > 0$ satisfying $\kappa\bar{p} \leq p$, we have $\int_0^\infty X_s^{\kappa\bar{p}}\mathrm{d}s = \infty$ almost surely.*

**Proof.** Let $\bar{a}_i(x) = a_i(x)/x^{\kappa\bar{p}}$ for $i = 1, 2, 3$. Then by the same argument as in [20, Theorem 2.15], there are, on an extended probability space, a Brownian motion $(\bar{B}_t)_{t\geq 0}$ and compensated Poisson random measure $\{\tilde{\bar{M}}(\mathrm{d}t, \mathrm{d}z, \mathrm{d}u)\}$ with intensity $\mathrm{d}t\mu(\mathrm{d}z)\mathrm{d}u$ so that there is a nonnegative process $(\bar{X}_t)_{t\geq 0}$ solving:

$$\bar{X}_t = \bar{X}_0 - \int_0^t \bar{a}_1(\bar{X}_s)\mathrm{d}s + \int_0^t \bar{a}_2(\bar{X}_s)^{1/2}\mathrm{d}\bar{B}_s + \int_0^t\int_0^\infty\int_0^{\bar{a}_3(\bar{X}_{s-})} z\tilde{\bar{M}}(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u).$$

Moreover, by [20, Proposition 2.16],

$$\int_0^\infty X_s^{\kappa\bar{p}}\mathrm{d}s = \tau_0^{\bar{X}} \tag{3.45}$$

almost surely. Recall the function $H_{1,0}$ in (1.4). Since $\kappa\bar{p} \leq p$, then under Condition 1.6(ia), for all $0 < u < 1$,

$$\bar{a}_1(u)u^{-1} + 2^{-1}\bar{a}_2(u)u^{-2} + \bar{a}_3(u)H_{1,0}(u) = G_{1,0}(u)u^{-\kappa\bar{p}} \leq G_{1,0}(u)u^{-p} \leq a.$$

Now by Lemma 3.2(ii), $\tau_0^{\bar{X}} = \infty$ almost surely. Then the assertion follows from (3.45) immediately. $\square$

**Proof of Theorem 1.7.** Let $\bar{p} \in (0, 1)$ satisfy $\bar{p}\kappa \leq p$. By Lemmas 3.5 and 3.2(i) and Condition 1.6(ic),

$$\int_0^\infty \kappa(X_s)^{\bar{p}}\mathrm{d}s = \infty$$

almost surely. Taking $\delta = \bar{p}$ in Lemma 3.4, we have $\mathbf{P}\{\tau_0^Y < \infty\} = 1$ for

$$\frac{q}{q+1-\theta} < \bar{p} \quad \text{and} \quad \bar{p}\kappa \leq p,$$

which finishes the proof. $\square$

### 3.5. *Proof of Theorem 1.8*

Different from proofs of the previous theorems, in Theorem 1.8 we adopt an approach similar to that in [20]. In the proof we first identify a power function $\hat{X}_t$ such that on one hand, with a positive probability $\hat{X}_t$ stays above $X_t$ for all large $t$, and on the other hand, $\hat{X}_t$ decreases faster enough for large $t$ so that if the term $\kappa(X_s)$ is replaced by $\kappa(\hat{X}_t)$ in the SDE for $Y$ in (1.1), then $Y$ becomes extinguishing with a strictly positive probability. To implement this idea we further approximate $\hat{X}_t$ by a step function and construct a process $\hat{Y}_t$ as a piecewise solution to a modified SDE for $Y$. The desired result then follows from a comparison theorem for SDE.

We state the following comparison theorem with its proof postponed to Appendix. For $i = 1, 2$ let $\{B_i(t, u) : t \geq 0, u \in \mathbb{R}\}$ be a two-parameter real-valued process with $(u, \omega) \mapsto B_i(t, u, \omega)$ measurable with respect to $\mathscr{B}(\mathbb{R}) \times \mathscr{F}_t$ for each $t \geq 0$. Let $U$ and $V$ be Borel functions on $\mathbb{R}$ and $V \geq 0$.

**Proposition 3.6.** *For $i = 1, 2$, let the càdlàg $\mathbb{R}$-valued process $(x_i(t))_{t\geq 0}$ be the solution to SDE*

$$x_i(t) = x_i(0) + \int_0^t B_i(s, x_i(s))\mathrm{d}s + \int_0^t U(x_i(s))\mathrm{d}W_s + \int_0^t \int_0^\infty \int_0^{V(x_i(s-))} z\tilde{N}(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u). \tag{3.46}$$

*Suppose that $B_1(t, u) \leq B_2(t, u)$ for all $t \geq 0$ and $u \in \mathbb{R}$. In addition, assume that $u \mapsto V(u)$ is nondecreasing and that there exist a sequence of increasing stopping times $(\gamma_n)_{n\geq 1}$ and a sequence of nonnegative constants $(C_n)_{n\geq 1}$ so that*

$$|B_1(s, u) - B_1(s, v)| + |U(u) - U(v)| + |V(u) - V(v)| \leq C_n|u - v|$$

*for all $n^{-1} \leq |u|, |v| \leq n$ and $s \leq \gamma_n$. If $\mathbf{P}\{x_1(0) \leq x_2(0)\} = 1$, then*

$$\mathbf{P}\{x_1(t) \leq x_2(t) \quad \text{for all} \quad 0 < t < \tilde{\gamma}\} = 1,$$

*where*

$$\tilde{\gamma} := \lim_{n\to\infty} \tilde{\gamma}_n \quad \text{and} \quad \tilde{\gamma}_n := \gamma_n \wedge \tau_{1/n}^{x_1} \wedge \tau_{1/n}^{x_2} \wedge \sigma_n^{x_1} \wedge \sigma_n^{x_2} \tag{3.47}$$

*with $\tau_{1/n}^{x_i} := \inf\{t \geq 0 : |x_i(t)| \leq 1/n\}$ and $\sigma_n^{x_i} := \inf\{t \geq 0 : |x_i(t)| \geq n\}$ for $i = 1, 2$.*

Recall the constant $c^*$ in Condition 1.6 and the definitions of stopping times $\tau_w^X = \tau^X(w)$ and $\sigma_w^X = \sigma^X(w)$ in (1.8) and (1.9) for constant $w > 0$.

**Lemma 3.7.** *Under Condition 1.6(iia) with $p > 0$, for any $0 < w < X_0$ and $0 < v \leq c^*$ we have*

$$\mathbf{E}\big[\tau_w^X \wedge \sigma_v^X\big] \leq 2p(p \wedge 1)^{-1}[1 - 2^{-1}(p \wedge 1)]^{-1}a^{-1}(X_0 - w)w^{-p-1}.$$

**Proof.** It is elementary to see that for $\delta > \delta_1 > 0$ and $u > 0$,

$$(1 + u)^{-\delta} - 1 + \delta u \geq (1 + u)^{-\delta_1} - 1 + \delta_1 u \geq -[(1 + u)^{\delta_1} - 1 - \delta_1 u] \geq 0,$$

which implies that for $p_1 := 2^{-1}(p \wedge 1)$,

$$p(p + 1)H_{1,p}(u) \geq p_1(1 - p_1)H_{1,-p_1}(u) \geq p_1(1 - p_1)H_{1,0}(u), \tag{3.48}$$

where the function $H_{1,p}$ and $H_{1,0}$ are defined in (1.2) and (1.4), respectively.

By (1.1) and Itô's formula,

$$\begin{aligned} X_t^{-p} = {} & X_0^{-p} + p\int_0^t a_1(X_s)X_s^{-p-1}\mathrm{d}s + \frac{p(p+1)}{2}\int_0^t a_2(X_s)X_s^{-p-2}\mathrm{d}s \\ & + p(p + 1)\int_0^t a_3(X_s)X_s^{-p}H_{1,p}(X_s)\mathrm{d}s - p\int_0^t a_2(X_s)^{1/2}X_s^{-p-1}\mathrm{d}B_s \\ & + \int_0^t \int_0^\infty \int_0^{a_3(X_{s-})} [(X_{s-} + z)^{-p} - X_{s-}^{-p}]\tilde{M}(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u). \end{aligned}$$

Since $X_t^{-p} \le w^{-p}$ for $0 \le t \le \tau_w^X$, then by (3.48), for $0 < p_1 < p \wedge 1$, we have

$$\begin{aligned}
w^{-p} &\ge \mathbf{E}\Big[X^{-p}_{t\wedge\tau_w^X\wedge\sigma_v^X}\Big] \\
&\ge X_0^{-p} + \mathbf{E}\Big[\int_0^{t\wedge\tau_w^X\wedge\sigma_v^X}\Big(pa_1(X_s)X_s^{-p-1} + \tfrac{p(p+1)}{2}a_2(X_s)X_s^{-p-2} \\
&\qquad\qquad + p(p+1)a_3(X_s)X_s^{-p}H_{1,p}(X_s)\Big)\,\mathrm{d}s\Big] \\
&\ge X_0^{-p} + p_1(1-p_1)\mathbf{E}\Big[\int_0^{t\wedge\tau_w^X\wedge\sigma_v^X} G_{1,0}(X_s)X_s^{-p}\mathrm{d}s\Big] \\
&\ge X_0^{-p} + p_1(1-p_1)a\mathbf{E}\big[t\wedge\tau_w^X\wedge\sigma_v^X\big],
\end{aligned}$$

where Condition 1.6(iia) is used in the last inequality. Using Fatou's lemma we get

$$\begin{aligned}
\mathbf{E}[\tau_w^X\wedge\sigma_v^X] &\le \liminf_{t\to\infty}\mathbf{E}\big[t\wedge\tau_w^X\wedge\sigma_v^X\big] \\
&\le (p_1(1-p_1)a)^{-1}(w^{-p} - X_0^{-p}) \\
&\le p(p_1(1-p_1)a)^{-1}(X_0 - w)w^{-p-1},
\end{aligned}$$

where we need the mean value theorem for the last inequality. This ends the proof. □

Since process $X$ turns to decrease in the long run, we next find a power function of time that is uniformly larger than $X_t$ for all large $t$ with a probability close to one. To this end, we consider a partition of the duration of time into consecutive time intervals with partition points increasing geometrically. Then for an arbitrary time interval in the partition, we further choose three levels $0 < l_1 < l_2 < l_3$ properly so that, during this time interval, process $X$ typically reaches level $l_1$ first before upcrossing level $l_3$, and then it typically stays below level $l_2$ continuously after having reached level $l_1$. The above choices of time partition and the associated levels allow us to show that process $(X_t)_{t\ge0}$ typically stay below the desired power function of $t$ for all large $t$.

**Lemma 3.8.** *Under Condition 1.6(iia) with $p > 0$, for any $\delta > 0$ and small enough $\varepsilon \in (0,1)$, there are constants $C(\delta,\varepsilon) > 0$ and $\delta_1 \in (0,1)$ that does not depend on $\varepsilon$ so that for $X_0 = \varepsilon^m$ with large enough $m \ge 1$, we have*

$$\mathbf{P}\{X_t \le t^{-\frac{1}{p+\delta}} \wedge c^* \text{ for all } t > 0\} \ge 1 - C(\delta,\varepsilon)\varepsilon^{m\delta_1/8}.$$

**Proof.** In the following let $\varepsilon_n := \varepsilon^n$ for $n \ge 1$. For any $\delta > 0$, let

$$\delta_1 := \frac{\delta}{2p+2+\delta} < 1. \tag{3.49}$$

For any fixed positive integer $m$ define

$$\bar{K}_m := \Big\{\sup_{t\le\varepsilon_m^{-p-(p+2)\delta_1}} X_t \le \varepsilon_m^{1-\delta_1},\ X_{\varepsilon_m^{-p-(p+2)\delta_1}} \le \varepsilon_{m+1}\Big\}$$

and

$$\bar{K}_n := \Big\{\sup_{\varepsilon_{n-1}^{-p-(p+2)\delta_1}\le t<\varepsilon_n^{-p-(p+2)\delta_1}} X_t \le \varepsilon_n^{1-\delta_1},\ X_{\varepsilon_n^{-p-(p+2)\delta_1}} \le \varepsilon_{n+1}\Big\}$$

for $n > m$. In the following we first show that $X_t \le t^{-1/(p+\delta)} \wedge c^*$ for all $t > 0$ on $\cap_{n=m}^\infty \bar{K}_n$.

It is obvious that $X_t \leq c^*$ for all $t \geq 0$ on event $\cap_{n=m}^{\infty} \bar{K}_n$ for $\varepsilon$ small enough. Let $r = \frac{1}{p+\delta}$. By (3.49), it is easy to check that

$$-\frac{1}{p+\delta} = -r = -\frac{1-\delta_1}{p+(p+2)\delta_1}. \tag{3.50}$$

Thus, for all $n \geq m$ we have $(\varepsilon_n^{-p-(p+2)\delta_1})^{-r} = \varepsilon_n^{1-\delta_1}$ by (3.50). Therefore, on event $\cap_{n=m}^{\infty} \bar{K}_n$, for any $t > 0$ with

$$\varepsilon_{n-1}^{-p-(p+2)\delta_1} \leq t < \varepsilon_n^{-p-(p+2)\delta_1}, \quad n \geq m+1, \tag{3.51}$$

we have

$$X_t \leq \varepsilon_n^{1-\delta_1} = \varepsilon_n^{-r(-p-(p+2)\delta_1)} < t^{-r}$$

by (3.50) again and (3.51), and for $0 \leq t \leq \varepsilon_m^{-p-(p+2)\delta_1}$ we also have

$$X_t \leq \varepsilon_m^{1-\delta_1} = \varepsilon_m^{-r(-p-(p+2)\delta_1)} \leq t^{-r}.$$

We now estimate the probability of $\cap_{n=m}^{\infty} \bar{K}_n$. In the rest of the proof we use notations

$$\mathbf{E}_{\bar{\varepsilon}}[\,\cdot\,] = \mathbf{E}\big[\cdot|X_0 = \bar{\varepsilon}\big] \text{ and } \mathbf{P}_{\bar{\varepsilon}}\{\,\cdot\,\} = \mathbf{P}\big\{\cdot|X_0 = \bar{\varepsilon}\big\}, \qquad \bar{\varepsilon} > 0.$$

By Lemma 3.7, there is a constant $c_1 > 0$ independent of $\varepsilon$ and $n$ so that

$$\mathbf{E}_{\bar{y}}\Big[\tau^X(\varepsilon_{n+1}^{1+\delta_1}) \wedge \sigma_{c^*}^X\Big] \leq c_1 a^{-1}(\varepsilon_n - \varepsilon_{n+1}^{1+\delta_1})\varepsilon_{n+1}^{(1+\delta_1)(-p-1)} \leq c_1 a^{-1}\varepsilon^{-1}\varepsilon_{n+1}^{-p-(p+1)\delta_1} \tag{3.52}$$

for all $0 < \bar{y} \leq \varepsilon_n$, where the constant $a > 0$ is determined in Condition 1.6(iia). Using the Markov inequality and (3.52) we obtain

$$\begin{aligned}\mathbf{P}_{\bar{y}}&\Big\{\tau^X(\varepsilon_{n+1}^{1+\delta_1}) \wedge \sigma_{c^*}^X > \varepsilon_n^{-p-(p+2)\delta_1} - \varepsilon_{n-1}^{-p-(p+2)\delta_1}\Big\}\\ &\leq [\varepsilon_n^{-p-(p+2)\delta_1} - \varepsilon_{n-1}^{-p-(p+2)\delta_1}]^{-1}\mathbf{E}_{\bar{y}}\big[\tau^X(\varepsilon_{n+1}^{1+\delta_1}) \wedge \sigma_{c^*}^X\big]\\ &\leq c_1 a^{-1}(1-\varepsilon^{p+(p+2)\delta_1})^{-1}\varepsilon^{-p-1+(n-p-1)\delta_1}\end{aligned} \tag{3.53}$$

for all $0 < \bar{y} \leq \varepsilon_n$. By Lemma 3.3, there is a constant $C > 0$ independent of $\varepsilon$ and $n$ so that

$$\mathbf{P}_{\bar{y}}\Big\{\sup_{t\geq 0} X_t \geq \varepsilon_n^{\delta_1-1}\Big\} \leq C(\varepsilon_n^{\delta_1-1}\bar{y})^{1/4} \leq C\varepsilon^{n\delta_1/4} \tag{3.54}$$

for all $0 < \bar{y} \leq \varepsilon_n$. By Fatou's lemma and (1.1),

$$\varepsilon_n \mathbf{P}_{\varepsilon_n^{1+\delta_1}}\big\{\sigma^X(\varepsilon_n) < \infty\big\} \leq \mathbf{E}_{\varepsilon_n^{1+\delta_1}}\big[X_{\sigma^X(\varepsilon_n)}\big] \leq \liminf_{t\to\infty} \mathbf{E}_{\varepsilon_n^{1+\delta_1}}\big[X_{t\wedge\sigma^X(\varepsilon_n)}\big] \leq \varepsilon_n^{1+\delta_1},$$

which implies

$$\mathbf{P}_{\varepsilon_n^{1+\delta_1}}\big\{\sigma^X(\varepsilon_n) < \infty\big\} \leq \varepsilon_n^{\delta_1}. \tag{3.55}$$

Similarly,

$$\mathbf{P}_{\bar{y}}\big\{\sigma_{c^*}^X < \infty\big\} \leq c^{*-1}\bar{y} \leq c^{*-1}\varepsilon_n \tag{3.56}$$

for all $\bar{y} \leq \varepsilon_n$.

To consider the complements of events $\bar{K}_m$ and $\bar{K}_n$, in the following, for the fixed $m \geq 1$, we introduce

$$E_m := \Big\{ \sup_{t \leq \varepsilon_m^{-p-(p+2)\delta_1}} X_t \geq \varepsilon_m^{1-\delta_1} \Big\} \bigcup \Big\{ \tau^X(\varepsilon_{m+1}^{1+\delta_1}) \wedge \sigma_{c^*}^X > \varepsilon_m^{-p-(p+2)\delta_1} \Big\}$$
$$\bigcup \Big\{ \tau^X(\varepsilon_{m+1}^{1+\delta_1}) \wedge \sigma_{c^*}^X < \infty, \sigma^X(\varepsilon_{m+1}) \circ \vartheta(\tau^X(\varepsilon_{m+1}^{1+\delta_1}) \wedge \sigma_{c^*}^X) \leq \varepsilon_m^{-p-(p+2)\delta_1} \Big\}$$

and for $n > m$,

$$E_n := \Big\{ \sup_{\varepsilon_{n-1}^{-p-(p+2)\delta_1} \leq t \leq \varepsilon_n^{-p-(p+2)\delta_1}} X_t \geq \varepsilon_n^{1-\delta_1} \Big\}$$
$$\bigcup \Big\{ \tau^X(\varepsilon_{n+1}^{1+\delta_1}) \wedge \sigma_{c^*}^X > \varepsilon_n^{-p-(p+2)\delta_1} - \varepsilon_{n-1}^{-p-(p+2)\delta_1} \Big\}$$
$$\bigcup \Big\{ \tau^X(\varepsilon_{n+1}^{1+\delta_1}) \wedge \sigma_{c^*}^X < \infty, \sigma_{\varepsilon_{n+1}}^X \circ \vartheta(\tau^X(\varepsilon_{n+1}^{1+\delta_1}) \wedge \sigma_{c^*}^X) \leq \varepsilon_n^{-p-(p+2)\delta_1} \Big\},$$

where $\vartheta(t)$ denotes the usual shift operator. For $0 < \bar{y} \leq \varepsilon_n$, we have

$$\begin{aligned}
&\mathbf{P}_{\bar{y}} \Big\{ \tau^X(\varepsilon_{n+1}^{1+\delta_1}) \wedge \sigma_{c^*}^X < \infty, \sigma^X(\varepsilon_{n+1}) \circ \vartheta(\tau^X(\varepsilon_{n+1}^{1+\delta_1}) \wedge \sigma_{c^*}^X) \leq \varepsilon_n^{-p-(p+2)\delta_1} \Big\} \\
&\leq \mathbf{P}_{\bar{y}}\{\sigma_{c^*}^X < \infty\} + \mathbf{P}_{\bar{y}} \Big\{ \tau^X(\varepsilon_{n+1}^{1+\delta_1}) < \infty, \sigma^X(\varepsilon_{n+1}) \circ \vartheta(\tau^X(\varepsilon_{n+1}^{1+\delta_1})) \leq \varepsilon_n^{-p-(p+2)\delta_1} \Big\} \\
&\leq \mathbf{P}_{\bar{y}}\{\sigma_{c^*}^X < \infty\} + \mathbf{P}_{\varepsilon_{n+1}^{1+\delta_1}} \Big\{ \sigma^X(\varepsilon_{n+1}) \leq \varepsilon_n^{-p-(p+2)\delta_1} \Big\} \\
&\leq \mathbf{P}_{\bar{y}}\{\sigma_{c^*}^X < \infty\} + \mathbf{P}_{\varepsilon_{n+1}^{1+\delta_1}} \Big\{ \sigma^X(\varepsilon_{n+1}) < \infty \Big\},
\end{aligned}$$

and then by (3.53)–(3.56),

$$\begin{aligned}
&\mathbf{P}_{\bar{y}}\{E_n\} \\
&\leq \mathbf{P}_{\bar{y}} \Big\{ \sup_{t \leq \varepsilon_n^{-p-(p+2)\delta_1}} X_t \geq \varepsilon_n^{1-\delta_1} \Big\} + \mathbf{P}_{\bar{y}} \Big\{ \tau^X(\varepsilon_{n+1}^{1+\delta_1}) \wedge \sigma_{c^*}^X > \varepsilon_n^{-p-(p+2)\delta_1} - \varepsilon_{n-1}^{-p-(p+2)\delta_1} \Big\} \\
&\quad + \mathbf{P}_{\bar{y}} \Big\{ \tau^X(\varepsilon_{n+1}^{1+\delta_1}) \wedge \sigma_{c^*}^X < \infty, \sigma^X(\varepsilon_{n+1}) \circ \vartheta(\tau^X(\varepsilon_{n+1}^{1+\delta_1}) \wedge \sigma_{c^*}^X) \leq \varepsilon_n^{-p-(p+2)\delta_1} \Big\} \\
&\leq \mathbf{P}_{\bar{y}} \Big\{ \sup_{t \leq \varepsilon_n^{-p-(p+2)\delta_1}} X_t \geq \varepsilon_n^{1-\delta_1} \Big\} + \mathbf{P}_{\bar{y}}\{\sigma_{c^*}^X < \infty\} + \mathbf{P}_{\varepsilon_{n+1}^{1+\delta_1}} \Big\{ \sigma^X(\varepsilon_{n+1}) < \infty \Big\} \\
&\quad + \mathbf{P}_{\bar{y}} \Big\{ \tau^X(\varepsilon_{n+1}^{1+\delta_1}) \wedge \sigma_{c^*}^X > \varepsilon_n^{-p-(p+2)\delta_1} - \varepsilon_{n-1}^{-p-(p+2)\delta_1} \Big\} \\
&\leq C\varepsilon_n^{\delta_1/4} + \varepsilon_n/c^* + \varepsilon_{n+1}^{\delta_1} + c_1 a^{-1}(1 - \varepsilon^{p+(p+2)\delta_1})^{-1} \varepsilon^{-p-1+(n-p-1)\delta_1} \\
&\leq \varepsilon^{n\delta_1/8}(1 + \varepsilon^{-(p+1)(\delta_1+1)}) =: M_n
\end{aligned}$$

for small enough $\varepsilon$. Similarly, for small enough $\varepsilon$ and $0 < \bar{y} \leq \varepsilon_m$,

$$\mathbf{P}_{\bar{y}}\{E_m\} \leq \varepsilon^{m\delta_1/8}(1 + \varepsilon^{-(p+1)(\delta_1+1)}) =: M_m.$$

Let $\bar{K}_n^c$ denote the complement of set $\bar{K}_n$. Note that $\bar{K}_n^c \subset E_n$. Then $\mathbf{P}_{\bar{y}}\{\bar{K}_n^c\} \leq M_n$ for all $n \geq m$ and $0 < \bar{y} \leq \varepsilon_n$. It follows that

$$\begin{aligned}
\mathbf{P}_{\varepsilon_m}\{\cup_{n=m}^{\infty} \bar{K}_n^c\} &= \mathbf{P}_{\varepsilon_m}\{\bar{K}_m^c\} + \sum_{n=m+1}^{\infty} \mathbf{P}_{\varepsilon_m}\{\cap_{i=m}^{n-1} \bar{K}_i \cap \bar{K}_n^c\} \\
&= \mathbf{P}_{\varepsilon_m}\{\bar{K}_m^c\} + \sum_{n=m+1}^{\infty} \mathbf{E}_{\varepsilon_m}\Big[ 1_{\cap_{i=m}^{n-1} \bar{K}_i} \mathbf{P}\Big\{ \bar{K}_n^c | X_{\varepsilon_{n-1}^{-p-(p+2)\delta_1}} \Big\} \Big]
\end{aligned}$$

$$\leq M_m + \sum_{n=m+1}^{\infty} M_n \mathbf{E}_{\varepsilon_m}\left[1_{\cap_{i=m}^{n-1}\bar{K}_i}\right]$$
$$\leq \sum_{n=m}^{\infty} M_n = (1-\varepsilon^{\delta_1/8})^{-1}(1+\varepsilon^{-(p+1)(1+\delta_1)})\varepsilon^{m\delta_1/8}.$$

Then

$$\mathbf{P}_{\varepsilon_m}\{\cap_{n=m}^{\infty}\bar{K}_n\} = 1 - \mathbf{P}_{\varepsilon_m}\{\cup_{n=m}^{\infty}\bar{K}_n^c\},$$

which finishes the proof. □

Using the estimate as function of time obtained in Lemma 3.8 we can construct a process $\hat{Y}$ which does not become extinct with a positive probability and can be shown by the comparison theorem to be uniformly smaller than process $Y$ with a probability close to one.

For small enough $\delta, \epsilon \in (0, c^* \wedge 1)$, suggested by Lemma 3.8 we define

$$\hat{X}(t) = t^{-\frac{1}{p+\delta}} \wedge c^* \tag{3.57}$$

and $\epsilon_n = \epsilon^{n^2}$. Let $(Y_1(t))_{t\geq 0}$ be the nonnegative solution to

$$Y_1(t) = Y_0 - \int_0^t \left[b_1(Y_1(s)) + \theta(Y_1(s))\hat{X}(0)^\kappa\right]\mathrm{d}s + \int_0^t b_2(Y_1(s))^{1/2}\mathrm{d}W_s + \int_0^t\int_0^\infty\int_0^{b_3(Y_1(s-))} z\tilde{N}(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u)$$

and $\gamma_1 := \inf\{t \geq 0 : Y_1(t) < \epsilon_1\}$. Define $\hat{Y}(t) := Y_1(t)$ for $t \in [0, \gamma_1]$. Suppose that $\hat{Y}(t)$ has been defined for $t \in [0, T_n]$ with $T_n := \sum_{i=1}^n \gamma_i$. Let $(Y_{n+1}(t))_{t\geq 0}$ be the nonnegative solution to

$$Y_{n+1}(t) = Y_n(T_n) - \int_0^t \left[b_1(Y_{n+1}(s)) + \theta(Y_{n+1}(s))\hat{X}(T_n)^\kappa\right]\mathrm{d}s + \int_0^t b_2(Y_{n+1}(s))^{1/2}\mathrm{d}W_s + \int_0^t\int_0^\infty\int_0^{b_3(Y_{n+1}(s-))} z\tilde{N}(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u) \tag{3.58}$$

and $\gamma_{n+1} := \inf\{t \geq 0 : Y_{n+1}(t) < \epsilon_{n+1}\}$. Define $\hat{Y}(t) := Y_{n+1}(t - T_n)$ for $t \in (T_n, T_n + \gamma_{n+1}] = (T_n, T_{n+1}]$. Then by the argument in [20, Theorem 3.1] and Condition 1.6(iii), $\hat{Y}$ is a piecewise time homogeneous spectrally positive Markov process.

Choose $l$ satisfying that

$$0 < l < q \quad \text{and} \quad \frac{l\kappa}{p+\delta} - l + \theta - 1 > 0. \tag{3.59}$$

Such a value $l$ exists if (1.11) holds and $\delta > 0$ is small enough. In the next lemma, we want to show that the process $\hat{Y}$ reaches 0 with a small probability.

**Lemma 3.9.** *Suppose that Condition 1.6(iib)–(iic) and condition (1.11) hold. For the constant $\delta$ in (3.59) and small $\epsilon > 1$, there is an integer $n_0 > 0$ so that*

$$\mathbf{P}\{\tau_0^{\hat{Y}} = \infty\} \geq A_0(\epsilon) := \prod_{n=n_0}^{\infty}(1 - 2\epsilon^{\delta(2n-1)} - \epsilon^{2n-3}).$$

**Proof.** In this proof we use $\mathbf{E}_{\bar{\varepsilon}}$ and $\mathbf{P}_{\bar{\varepsilon}}$ to denote the conditional expectation and conditional probability with respect to $\mathscr{F}_{\hat{Y}(\tau^{\hat{Y}}_{\bar{\varepsilon}})}$. We first estimate $\mathbf{P}_{\epsilon_n}\{\gamma_{n+1} > \epsilon_{n+1}^{-l} | \gamma_n > \epsilon_n^{-l}\}$. Recall that $\sigma^{Y_{n+1}}(\epsilon_{n-1}) := \inf\{t \geq 0 : Y_{n+1}(t) > \epsilon_{n-1}\}$. By Fatou's lemma and (3.58),

$$\begin{aligned}
\epsilon_{n-1}\mathbf{P}_{\epsilon_n}&\left\{\sigma^{Y_{n+1}}(\epsilon_{n-1}) < \gamma_{n+1}\big|\gamma_n > \epsilon_n^{-l}\right\} \\
&\leq \mathbf{E}_{\epsilon_n}\Big[Y_{n+1}(\sigma^{Y_{n+1}}(\epsilon_{n-1}) \wedge \gamma_{n+1})1_{\{\sigma^{Y_{n+1}}(\epsilon_{n-1})<\gamma_{n+1}\}}\big|\gamma_n > \epsilon_n^{-l}\Big] \\
&\leq \mathbf{E}_{\epsilon_n}\Big[Y_{n+1}(\sigma^{Y_{n+1}}(\epsilon_{n-1}) \wedge \gamma_{n+1})\big|\gamma_n > \epsilon_n^{-l}\Big] \\
&\leq \liminf_{t\to\infty}\mathbf{E}_{\epsilon_n}\Big[Y_{n+1}(t \wedge \sigma^{Y_{n+1}}(\epsilon_{n-1}) \wedge \gamma_{n+1})\big|\gamma_n > \epsilon_n^{-l}\Big] \leq \epsilon_n,
\end{aligned}$$

which implies

$$\mathbf{P}_{\epsilon_n}\left\{\sigma^{Y_{n+1}}(\epsilon_{n-1}) < \gamma_{n+1}\big|\gamma_n > \epsilon_n^{-l}\right\} \leq \epsilon_n/\epsilon_{n-1} = \epsilon^{2n-1}. \tag{3.60}$$

Note that by (3.57),

$$\hat{X}(T_n) \leq \gamma_n^{-\frac{1}{p+\delta}} < \epsilon_n^{\frac{l}{p+\delta}} \quad \text{given } \gamma_n > \epsilon_n^{-l} \tag{3.61}$$

for all $n \geq 1$.

For $\delta > 0$, by (3.58), the definition of the process $(Y_n(t))_{t\geq 0}$ and Itô's formula, with respect to $\{\mathscr{F}_{Y_n(\tau^{Y_n}_{\epsilon_n})}\}$ and for $\gamma_n > \epsilon_n^{-l}$,

$$t \mapsto Y_{n+1}(t \wedge \gamma_{n+1})^{-\delta}\exp\Big\{-\int_0^{t\wedge\gamma_{n+1}} G_\delta(\hat{X}(T_n), Y_{n+1}(s))\mathrm{d}s\Big\}$$

is a martingale, where

$$G_\delta(u, v) := \delta[b_1(v) + \theta(v)u^\kappa]v^{-1} + \frac{\delta(\delta+1)}{2}b_2(v)v^{-2} + \delta(\delta+1)b_3(v)H_{2,\delta}(v)$$

with the function $H_{2,\delta}$ defined in (1.3). Taking an expectation and using Fatou's lemma, for all $n \geq 1$, we have

$$\begin{aligned}
\epsilon_n^{-\delta} &= \liminf_{t\to\infty}\mathbf{E}_{\epsilon_n}\Big[Y_{n+1}(t\wedge\gamma_{n+1})^{-\delta}\exp\Big\{-\int_0^{t\wedge\gamma_{n+1}} G_\delta(\hat{X}(T_n), Y_{n+1}(s))\mathrm{d}s\Big\}\big|\gamma_n > \epsilon_n^{-l}\Big] \\
&\geq \mathbf{E}_{\epsilon_n}\Big[\liminf_{t\to\infty} Y_{n+1}(t\wedge\gamma_{n+1})^{-\delta}\exp\Big\{-\int_0^{t\wedge\gamma_{n+1}} G_\delta(\hat{X}(T_n), Y_{n+1}(s))\mathrm{d}s\Big\}\big|\gamma_n > \epsilon_n^{-l}\Big] \\
&= \epsilon_{n+1}^{-\delta}\mathbf{E}_{\epsilon_n}\Big[\exp\Big\{-\int_0^{\gamma_{n+1}} G_\delta(\hat{X}(T_n), Y_{n+1}(s))\mathrm{d}s\Big\}\big|\gamma_n > \epsilon_n^{-l}\Big].
\end{aligned} \tag{3.62}$$

Observe that under (3.59), there is a constant $n_0 > 1$ so that for all $n \geq n_0$,

$$\begin{aligned}
\delta_0(n) &:= (n+1)^2(-l+\theta-1) + n^2\frac{l\kappa}{p+\delta} \\
&= (n+1)^2[\frac{l\kappa}{p+\delta} - l + \theta - 1] - (2n+1)\frac{l\kappa}{p+\delta} > 0
\end{aligned} \tag{3.63}$$

and

$$\delta_1(n) := -l(n+1)^2 + q(n-1)^2 = (q-l)(n-1)^2 - 4nl > 0. \tag{3.64}$$

Under Condition 1.6(iib) and (iic), for all $0 < u, v \leq c^*$, we have

$$G_\delta(u, v) \leq \delta c_\theta u^\kappa v^{\theta-1} + \delta(\delta+1)G_{2,0}(v) \leq \delta(\delta+1)(c_\theta \vee b)[u^\kappa v^{\theta-1} + v^q]$$

with $G_{2,0}$ given in (1.7) and

$$\epsilon_{n+1} \le Y_{n+1}(s) \le \epsilon_{n-1} \quad \text{for} \quad s < \gamma_{n+1} \wedge \sigma^{Y_{n+1}}(\epsilon_{n-1}).$$

Then by (3.61), given $\gamma_n > \epsilon_n^{-l}$ and for $s < \gamma_{n+1} \wedge \sigma^{Y_{n+1}}(\epsilon_{n-1})$,

$$\begin{aligned} G_\delta(\hat{X}(T_n), Y_{n+1}(s)) &\le \delta(\delta+1)(c_\theta \vee b)[\hat{X}(T_n)^\kappa Y_{n+1}(s)^{\theta-1} + Y_{n+1}(s)^q] \\ &\le \delta(\delta+1)(c_\theta \vee b)[\epsilon_n^{\frac{\kappa l}{p+\delta}} \epsilon_{n+1}^{\theta-1} + \epsilon_{n-1}^q]. \end{aligned}$$

It follows from (3.63) and (3.64) that, given $\gamma_n > \epsilon_n^{-l}$ and $\gamma_{n+1} < \epsilon_{n+1}^{-l} \wedge \sigma^{Y_{n+1}}(\epsilon_{n-1})$,

$$\begin{aligned} &\int_0^{\gamma_{n+1}} G_\delta(\hat{X}(T_n), Y_{n+1}(s))\mathrm{d}s \\ &\quad \le \delta(\delta+1)(c_\theta \vee b)\gamma_{n+1}[\epsilon_n^{\frac{\kappa l}{p+\delta}} \epsilon_{n+1}^{\theta-1} + \epsilon_{n-1}^q] \\ &\quad \le \delta(\delta+1)(c_\theta \vee b)\epsilon_{n+1}^{-l}\big(\epsilon_n^{\frac{\kappa l}{p+\delta}} \epsilon_{n+1}^{\theta-1} + \epsilon_{n-1}^q\big) \\ &\quad = \delta(\delta+1)(c_\theta \vee b)[\epsilon^{\delta_1(n)} + \epsilon^{\delta_0(n)}] \le \ln 2 \end{aligned} \tag{3.65}$$

for all $n \ge n_0$ and small enough $\epsilon$. From (3.62) and (3.65) it follows that all $n \ge n_0$ and small enough $\epsilon$,

$$\begin{aligned} \epsilon_n^{-\delta} &\ge \epsilon_{n+1}^{-\delta}\mathbf{E}_{\epsilon_n}\Big[\exp\Big\{-\int_0^{\gamma_{n+1}} G_\delta(\hat{X}(T_n), Y_{n+1}(s))\mathrm{d}s\Big\}1_{\{\gamma_{n+1}<\epsilon_{n+1}^{-l}\wedge\sigma^{Y_{n+1}}(\epsilon_{n-1})\}}\Big|\gamma_n > \epsilon_n^{-l}\Big] \\ &\ge 2^{-1}\epsilon_{n+1}^{-\delta}\mathbf{P}_{\epsilon_n}\big\{\gamma_{n+1} < \epsilon_{n+1}^{-l} \wedge \sigma^{Y_{n+1}}(\epsilon_{n-1})\big|\gamma_n > \epsilon_n^{-l}\big\} \\ &\ge 2^{-1}\epsilon_{n+1}^{-\delta}\Big[\mathbf{P}_{\epsilon_n}\big\{\gamma_{n+1} < \epsilon_{n+1}^{-l}\big|\gamma_n > \epsilon_n^{-l}\big\} - \mathbf{P}_{\epsilon_n}\big\{\gamma_{n+1} > \sigma^{Y_{n+1}}(\epsilon_{n-1})\big|\gamma_n > \epsilon_n^{-l}\big\}\Big]. \end{aligned}$$

It follows from (3.60) that for all $n \ge n_0$ and small enough $\epsilon$,

$$\begin{aligned} \mathbf{P}_{\epsilon_n}\big\{\gamma_{n+1} < \epsilon_{n+1}^{-l}\big|\gamma_n > \epsilon_n^{-l}\big\} &\le 2\epsilon_n^{-\delta}\epsilon_{n+1}^{\delta} + \mathbf{P}_{\epsilon^n}\big\{\gamma_{n+1} > \sigma^{Y_{n+1}}(\epsilon_{n-1})\big|\gamma_n > \epsilon_n^{-l}\big\} \\ &\le 2\epsilon^{\delta(2n+1)} + \epsilon^{2n-1}. \end{aligned}$$

Observe that by the Markov property,

$$\begin{aligned} \mathbf{P}\{\cap_{n=n_0}^m\{\gamma_n > \epsilon_n^{-l}\}\} &= \mathbf{E}\Big[\mathbf{P}_{\epsilon_{m-1}}\big\{\gamma_m > \epsilon_m^{-l} \mid \cap_{n=n_0}^{m-1}\{\gamma_n > \epsilon_n^{-l}\}\big\}1_{\cap_{n=n_0}^{m-1}\{\gamma_n > \epsilon_n^{-l}\}}\Big] \\ &= \mathbf{E}\Big[\mathbf{P}_{\epsilon_{m-1}}\big\{\gamma_m > \epsilon_m^{-l}|\gamma_{m-1} > \epsilon_{m-1}^{-l}\big\}1_{\cap_{n=n_0}^{m-1}\{\gamma_n > \epsilon_n^{-l}\}}\Big] \\ &\ge (1 - 2\epsilon^{\delta(2m-1)} - \epsilon^{2m-3})\mathbf{P}\{\cap_{n=n_0}^{m-1}\{\gamma_n > \epsilon_n^{-l}\}\} \\ &\ge \prod_{n=n_0}^m (1 - 2\epsilon^{\delta(2n-1)} - \epsilon^{2n-3}). \end{aligned}$$

Letting $m \to \infty$ we get

$$A_0(\epsilon) \le \mathbf{P}\big\{\cap_{n=n_0}^\infty\{\gamma_n > \epsilon_n^{-l}\}\big\} \le \mathbf{P}\{\tau_0^{\hat{Y}} = \infty\},$$

which ends the proof. $\square$

**Lemma 3.10.** *Under Condition 1.6(iia) with $p > 0$ and (iic) and (iii), for each $\delta > 0$ and small enough $\varepsilon > 0$, there are constants $C(\delta, \varepsilon) > 0$ and $\delta_1 \in (0, 1)$ that do not depend on $\varepsilon$ so that for all $X_0 = \varepsilon^m$ with large enough $m$ we have*

$$\mathbf{P}\{Y_t \ge \hat{Y}(t) \text{ for all } t \ge 0\} =: \mathbf{P}\{B\} \ge 1 - C(\delta, \varepsilon)\varepsilon^{m\delta_1/8}. \tag{3.66}$$

**Proof.** By Lemma 3.8, there are constants $C(\delta, \varepsilon) > 0$ and $\delta_1 \in (0, 1)$ so that for all $X_0 = \varepsilon^m$ with large $m$ we have

$$\mathbf{P}\{X_t \leq \hat{X}(t) \text{ for all } t \geq 0\} =: \mathbf{P}(A) \geq 1 - C(\delta, \varepsilon)\varepsilon^{m\delta_1/8}.$$

Observe that under Condition 1.6(iic), given $A$ and $s \geq T$, we have

$$\kappa(X_s) \leq X_s^\kappa \leq \hat{X}(s)^\kappa \leq \hat{X}(T)^\kappa.$$

Since $(B_t)_{t\geq 0}$, $(W_t)_{t\geq 0}$, $\{\tilde{M}(\mathrm{d}t, \mathrm{d}z, \mathrm{d}u)\}$ and $\{\tilde{N}(\mathrm{d}t, \mathrm{d}z, \mathrm{d}u)\}$ are independent, then by using (3.58), the definition of $(\hat{Y}(t))_{t\geq 0}$, Condition 1.6(iii) and Proposition 3.6, $\mathbf{P}\{B|A\} = 1$. It follows that

$$\mathbf{P}\{B\} \geq \mathbf{P}\{B|A\} \cdot \mathbf{P}\{A\} \geq 1 - C(\delta, \varepsilon)\varepsilon^{m\delta_1/8},$$

which ends the proof. □

**Lemma 3.11.** *Under Condition 1.6(iii), for each $\varepsilon > 0$, there is a constant $t_0 > 0$ so that*

$$\mathbf{P}\Big\{\sup_{t\geq t_0} X_t \leq \varepsilon, Y_{t_0} > 0\Big\} > 0. \tag{3.67}$$

**Proof.** Since $X_t \to 0$ as $t \to \infty$ by Lemma 3.2(i), there are constants $t_0 > 0$ and $n \geq 1$ so that

$$\mathbf{P}\Big\{\sup_{t\geq t_0} X_t \leq \varepsilon, \sup_{0\leq t<t_0} X_t \leq n\Big\} > 0. \tag{3.68}$$

Let $(\tilde{Y}_t)_{t\geq 0}$ be the nonnegative solution to

$$\begin{aligned}\tilde{Y}_t =& Y_0 - \int_0^t [b_1(\tilde{Y}_s) + C_n\theta(\tilde{Y}_s)]\mathrm{d}s + \int_0^t b_2(\tilde{Y}_s)^{1/2}\mathrm{d}W_s \\ &+ \int_0^t \int_0^\infty \int_0^{b_3(\tilde{Y}_{s-})} z\tilde{N}(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u),\end{aligned} \tag{3.69}$$

where $C_n := \sup_{x\in[0,n]} \kappa(x)$. Under Condition 1.6(iii), by the comparison theorem (Proposition 3.6),

$$\mathbf{P}\Big\{Y_t \geq \tilde{Y}_t \text{ for all } 0 \leq t \leq t_0 \Big| \sup_{0\leq t\leq t_0} X_t \leq n\Big\} = 1. \tag{3.70}$$

It is easy to see that $\mathbf{P}\{\tilde{Y}_{t_0} > 0\} > 0$. Since $(\tilde{Y}_t)_{t\geq 0}$ and $(X_t)_{t\geq 0}$ are independent, then by (3.68) we get

$$\mathbf{P}\Big\{\sup_{t\geq t_0} X_t \leq \varepsilon, \sup_{0\leq t<t_0} X_t \leq n, \tilde{Y}_{t_0} > 0\Big\} = \mathbf{P}\Big\{\sup_{t\geq t_0} X_t \leq \varepsilon, \sup_{0\leq t<t_0} X_t \leq n\Big\}\mathbf{P}\{\tilde{Y}_{t_0} > 0\} > 0.$$

Using (3.70) we get

$$\mathbf{P}\Big\{\sup_{t\geq t_0} X_t \leq \varepsilon, \sup_{0\leq t<t_0} X_t \leq n, Y_{t_0} > 0\Big\} > 0,$$

which implies (3.67). □

**Proof of Theorem 1.8.** We first show that for given $X_0 = \varepsilon^m$ and $Y_0$ with $m$ large enough and $\varepsilon$ small enough, there is a constant $C(\varepsilon) > 0$ so that

$$\mathbf{P}\{\tau_0^Y = \infty\} \geq C(\varepsilon). \tag{3.71}$$

Let $B^c$ denote the complementary set of $B$, which is given in (3.66). By Lemma 3.10, there are constants $C(\delta,\varepsilon)>0$ and $\delta_1\in(0,1)$ independent of $\delta$ so that

$$\mathbf{P}\{B^c\}\le C(\delta,\varepsilon)\varepsilon^{m\delta_1/8}. \tag{3.72}$$

Observe that

$$\begin{aligned}\{\tau_0^{\hat{Y}}=\infty\}&=(\{\tau_0^{\hat{Y}}=\infty\}\cap B)\cup(\{\tau_0^{\hat{Y}}=\infty\}\cap B^c)\\&\subset(\{\tau_0^{Y}=\infty\}\cap B)\cup B^c\subset\{\tau_0^{Y}=\infty\}\cup B^c.\end{aligned}$$

Therefore, by Lemma 3.9 and (3.72),

$$\begin{aligned}\mathbf{P}\{\tau_0^{Y}=\infty\}&\ge\mathbf{P}\{\{\tau_0^{Y}=\infty\}\cup B^c\}-\mathbf{P}\{B^c\}\\&\ge\mathbf{P}\{\tau_0^{\hat{Y}}=\infty\}-\mathbf{P}\{B^c\}\ge A_0(\epsilon)-C(\delta,\varepsilon)\varepsilon^{m\delta_1/8}>0\end{aligned}$$

for $m$ large enough and small enough $\epsilon$ and $\varepsilon$, which gives (3.71) for some constant $C(\varepsilon)>0$.

By Lemma 3.11, for each $\varepsilon>0$, there is a constant $t_0:=t_0(\varepsilon)>0$ so that $\mathbf{P}\{X_{t_0}\le\varepsilon, Y_{t_0}>0\}>0$. By the Markov property and (3.71), for each $t>0$ and small enough $\varepsilon>0$, there is a constant $C(\varepsilon)>0$ so that for $X_t\le\varepsilon$ and $Y_t>0$, we have

$$\mathbf{P}\{\tau_0^{Y}=\infty|(X_t,Y_t)\}\ge C(\varepsilon).$$

It follows that

$$\mathbf{P}\big\{\tau_0^{Y}=\infty\big\}=\mathbf{P}\big\{X_{t_0}\le\varepsilon, Y_{t_0}>0\big\}\cdot\mathbf{P}\big\{\tau_0^{Y}=\infty|X_{t_0}\le\varepsilon, Y_{t_0}>0\big\}>0,$$

which ends the proof. □

### 3.6. *Proof of Theorem 1.9*

For $\delta\in(-1,0)\cup(0,\infty)$, and $i=1,2$ recall the definitions of $H_{i,\delta}$, $H_{i,0}$, $G_{i,0}$ in (1.2)–(1.7). For $x,y,\beta>0$ and $r\in(-(\beta^{-1}\wedge1),1)$ define

$$G_r(x,y):=\beta G_{1,r}(x)-G_{2,r}(y)-\kappa(x)\theta(y)y^{-1} \tag{3.73}$$

with

$$G_{1,r}(x):=a_1(x)x^{-1}+(1+\beta r)2^{-1}a_2(x)x^{-2}+(1+\beta r)a_3(x)H_{1,\beta r}(x) \tag{3.74}$$

and

$$G_{2,r}(y):=b_1(y)y^{-1}+(1-r)2^{-1}b_2(y)y^{-2}+(1-r)b_3(y)H_{2,-r}(y). \tag{3.75}$$

Thus,

$$G_0(x,y)=\beta G_{1,0}(x)-G_{2,0}(y)-\kappa(x)\theta(y)y^{-1}. \tag{3.76}$$

To prove Theorem 1.9, we first prove the following assertions.

**Lemma 3.12.** *For any $x,y>0$, we have*

$$G_r(x,y)\le(1-r)G_0(x,y)+\beta(\beta+1)rG_{1,0}(x),\qquad r\in(0,1) \tag{3.77}$$

*and*

$$G_r(x,y)\ge(1+\beta r)G_0(x,y)+(\beta+1)rG_{2,0}(y)+\beta r\kappa(x)\theta(y)y^{-1},\ \ r\in(-(\beta^{-1}\wedge1),0). \tag{3.78}$$

**Proof.** Observe that for each $i = 1, 2$, $x > 0$ and $r \geq 0$, we have

$$H_{i,r}(x) \leq H_{i,0}(x), \quad H_{i,-r}(x) \geq H_{i,0}(x),$$

which implies that

$$G_{1,r}(x) \leq (1 + \beta r)G_{1,0}(x), \quad G_{2,r}(x) \geq (1 - r)G_{2,0}(x).$$

Then for $x, y > 0$,

$$\begin{aligned} G_r(x, y) &\leq \beta(1 + \beta r)G_{1,0}(x) - (1 - r)G_{2,0}(y) - (1 - r)\kappa(x)\theta(y)y^{-1} \\ &= (1 - r)G_0(x, y) + \beta(\beta + 1)rG_{1,0}(x), \quad r \in (0, 1) \end{aligned}$$

and

$$\begin{aligned} G_r(x, y) &\geq \beta(1 + \beta r)G_{1,0}(x) - (1 - r)G_{2,0}(y) - \kappa(x)\theta(y)y^{-1} \\ &= (1 + \beta r)G_0(x, y) + (\beta + 1)rG_{2,0}(y) + \beta r\kappa(x)\theta(y)y^{-1}, \\ &\qquad r \in (-(\beta^{-1} \wedge 1), 0), \end{aligned}$$

which finishes the proof. □

The following result is key to the proof of Theorem 1.9.

**Lemma 3.13.** *Under the assumptions of Theorem 1.9, for any $0 < \varepsilon_1 < \varepsilon$, if $X_0, Y_0 \leq \varepsilon_1$, then we have*

$$\mathbf{P}\{\tau_0^Y \wedge \sigma_\varepsilon^X \wedge \sigma_\varepsilon^Y < \infty\} = 1. \tag{3.79}$$

**Proof.** The proof is an application of Corollary 2.3. We first present the key function $g$ satisfying the conditions of Corollary 2.3. Define $g(u) := \mathrm{e}^{-\lambda u^r}$ for $u, \lambda > 0$ and $0 < r < 1$. Let $0 < \varepsilon < c^*$ (determined in Condition 1.6(i)) and $g(x, y) := g(x^{-\beta}y)$ for all $x, y > 0$, where the value of constant $\beta > 0$ is to be specified later. In the following we show that there are constants $d_1, d_2 > 0$ so that for all $0 < x, y < \varepsilon$, we have, respectively,

$$\begin{aligned} &Lg(x, y) \geq r\lambda d_1 g(x, y) \text{ under condition (i) of Theorem 1.9} \\ &\text{and } Lg(x, y) \geq r\lambda d_2 x^p g(x, y) \text{ under condition (ii) of Theorem 1.9.} \end{aligned} \tag{3.80}$$

Recall the definitions of $K_z^1$ and $K_z^2$ in (3.1) and $G_r, G_{1,r}, G_{2,r}$ in (3.73)–(3.75). For simplicity we denote $u = x^{-\beta}y$ in the following. By (3.1) and (3.7)–(3.8),

$$g(u)^{-1}K_z^1 g(x, y) \geq -r\beta(r\beta + 1)\lambda u^r z^2 x^{-2}\int_0^1 (1 + zx^{-1}v)^{-r\beta-2}(1 - v)\mathrm{d}v$$

and

$$g(u)^{-1}K_z^2 g(x, y) \geq r(1 - r)\lambda u^r z^2 y^{-2}\int_0^1 (1 + zy^{-1}v)^{r-2}(1 - v)\mathrm{d}v.$$

Then one can get

$$\begin{aligned} L_1 g(x, y) &= -\lambda r\beta g(u)u^r a_1(x)/x + 2^{-1}\big[(\lambda r\beta)^2 g(u)u^{2r} \\ &\qquad -\lambda r\beta(1 + r\beta)g(u)u^r\big]a_2(x)/x^2 + a_3(x)\int_0^\infty K_z^1 g(x, y)\mu(\mathrm{d}z) \\ &\geq -\lambda r\beta g(u)u^r G_{1,r}(x) \end{aligned}$$

and

$$\begin{aligned}L_2 g(x,y) &= \lambda r g(u)u^r[b_1(y)+\kappa(x)\theta(y)]/y + 2^{-1}\big[(\lambda r)^2 g(u)u^{2r} \\ &\quad +\lambda r(1-r)g(u)u^r\big]b_2(y)/y^2 + b_3(x)\int_0^\infty K_z^2 g(x,y)\nu(\mathrm{d}z) \\ &\geq \lambda r g(u)u^r[G_{2,r}(y)+\kappa(x)\theta(y)y^{-1}].\end{aligned}$$

Thus,

$$Lg(x,y) = L_1 g(x,y) + L_2 g(x,y) \geq -\lambda r u^r g(u) G_r(x,y) = -\lambda r u^r G_r(x,y) g(x,y). \tag{3.81}$$

In the following we use the inequality (3.77) in Lemma 3.12 to estimate $G_r$. Recall the definition of $G_0$ in (3.76).

Under condition (i) of Theorem 1.9, taking $\beta = \kappa/(1-\theta)$, we have $-\beta a + b > 0$. Then there exist small constants $c_1 > 0$ and $0 < r < 1-\theta$ so that

$$(1-r)(-\beta a + b) - a\beta(\beta+1)r \geq c_1 \tag{3.82}$$

Under condition (i) of Theorem 1.9 we have $p = q = 0$ and then using Condition 1.6(i) we obtain

$$\begin{aligned}-G_0(x,y) &= -\beta G_{1,0}(x) + G_{2,0}(y) + \kappa(x)\theta(y)y^{-1} \\ &\geq -\beta a + b + c_\theta x^\kappa y^{\theta-1} = -\beta a + b + c_\theta u^{\theta-1}\end{aligned} \tag{3.83}$$

and $G_{1,0}(x) \leq a$ for all $0 < x, y < \varepsilon$. It thus follows from (3.77) and (3.82)–(3.83) that

$$\begin{aligned}-G_r(x,y) &\geq -(1-r)G_0(x,y) - \beta(\beta+1)rG_{1,0}(x) \\ &\geq (1-r)(-\beta a + b) + (1-r)c_\theta u^{\theta-1} - a\beta(\beta+1)r \\ &\geq c_1 + (1-r)c_\theta u^{\theta-1}\end{aligned}$$

for all $0 < x, y < \varepsilon$. It then follows from Lemma 3.1 that

$$-u^r G_r(x,y) \geq c_1 u^r + (1-r)c_\theta u^{r+\theta-1} \geq d_1, \qquad 0 < x, y < \varepsilon$$

for some constant $d_1 > 0$. Then the first part of (3.80) follows from (3.81).

Under condition (ii) of Theorem 1.9 and Condition 1.6(i), taking $\beta = p/q$, we have $p = \beta q = \kappa - \beta(1-\theta)$ and then

$$\begin{aligned}-G_0(x,y) &\geq -\beta a x^p + b y^q + c_\theta x^\kappa y^{\theta-1} \\ &= x^p[-\beta a + b x^{-p} y^q + c_\theta x^{\kappa-p} y^{\theta-1}] \\ &= x^p[-\beta a + b u^q + c_\theta u^{\theta-1}], \quad 0 < x, y < \varepsilon.\end{aligned} \tag{3.84}$$

For

$$\bar{p} := 1 + q/(1-\theta) = [q+(1-\theta)](1-\theta)^{-1}, \quad \bar{q} := \bar{p}/(\bar{p}-1) = [q+(1-\theta)]q^{-1},$$

we have $q/\bar{p} + (\theta-1)/\bar{q} = 0$ and then by Lemma 3.1,

$$\begin{aligned}bu^q + c_\theta u^{\theta-1} &\geq \bar{p}^{1/\bar{p}}\bar{q}^{1/\bar{q}} b^{1/\bar{p}} c_\theta^{1/\bar{q}} u^{q/\bar{p}+(\theta-1)/\bar{q}} = \bar{p}^{1/\bar{p}}\bar{q}^{1/\bar{q}} b^{1/\bar{p}} c_\theta^{1/\bar{q}} \\ &= [q+(1-\theta)]\Big(\frac{b}{1-\theta}\Big)^{\frac{1-\theta}{q+1-\theta}}\Big(\frac{c_\theta}{q}\Big)^{\frac{q}{q+1-\theta}} =: c_2.\end{aligned} \tag{3.85}$$

Under condition (1.12), we have $c_2 > \beta a$. It follows from (3.84) that

$$-G_0(x,y)x^{-p} \geq -\beta a + bu^q + c_\theta u^{\theta-1} \geq c_2 - \beta a > 0, \quad 0 < x, y < \varepsilon.$$

Then by (3.77) and Condition 1.6(i) again, there are constants $0 < r < 1-\theta$ and $c_3 := (c_2 - \beta a)(1-r)$ so that $c_3 > r\beta(\beta+1)a$ and

$$\begin{aligned} -G_r(x,y)x^{-p} &\geq -(1-r)G_0(x,y)x^{-p} - r\beta(\beta+1)G_{1,0}(x)x^{-p} \\ &\geq c_3 - r\beta(\beta+1)a > 0 \end{aligned} \tag{3.86}$$

for all $0 < x, y < \varepsilon$. Then

$$-G_r(x,y)u^r \geq [c_3 - r\beta(\beta+1)a]x^p > 0, \qquad 0 < x, y < \varepsilon,\ u \geq 1. \tag{3.87}$$

Let $\delta > 0$ be a constant satisfying

$$((1-\delta)c_2 - \beta a)(1-r) - r\beta(\beta+1)a > 0. \tag{3.88}$$

By (3.84) and (3.85) we obtain

$$\begin{aligned} -G_0(x,y) &\geq x^p\big[-\beta a + (1-\delta)(bu^q + c_\theta u^{\theta-1}) + \delta(bu^q + c_\theta u^{\theta-1})\big] \\ &\geq x^p\big[((1-\delta)c_2 - \beta a) + \delta c_\theta u^{\theta-1}\big], \quad 0 < x, y < \varepsilon \end{aligned}$$

and then by (3.88) and the same argument as in (3.86),

$$\begin{aligned} -G_r(x,y)u^r &\geq x^p u^r\big[((1-\delta)c_2 - \beta a)(1-r) - r\beta(\beta+1)a + \delta c_\theta u^{\theta-1}\big] \\ &\geq \delta c_\theta x^p u^{r+\theta-1} \geq \delta c_\theta x^p, \qquad 0 < x, y < \varepsilon,\ u \leq 1. \end{aligned}$$

This and (3.87) imply the second part of (3.80) by (3.81).

Letting $\kappa\bar{p} = p$ in Lemma 3.5, we have $\int_0^\infty X_s^p \mathrm{d}s = \infty$ almost surely. Since $X_t \to 0$ as $t \to \infty$ by Lemma 3.2(i), then for all $0 < \varepsilon < c^*$, we have $\int_0^\infty (X_s^p \wedge \varepsilon)\mathrm{d}s = \infty$ almost surely. Using the above assertions and Corollary 2.3, we have

$$\mathbf{P}\{\tau_0^X \wedge \tau_0^Y \wedge \sigma_\varepsilon^X \wedge \sigma_\varepsilon^Y < \infty\} \geq \mathrm{e}^{-\lambda(X_0^{-\beta}Y_0)^r}$$

for all $0 < \varepsilon_1 < \varepsilon$ and $\lambda > 0$. Since $\tau_0^Y = \infty$ almost surely by Lemma 3.2(ii), then letting $\lambda \to 0$ in the above inequality we have (3.79). □

Now we are ready to prove Theorem 1.9.

**Proof of Theorem 1.9.** For any $0 < \varepsilon_1 < \varepsilon$, if $X_0, Y_0 \leq \varepsilon_1$, by Lemma 3.3 we have

$$\mathbf{P}\{\sigma_\varepsilon^X < \infty\} = \mathbf{P}\big\{\sup_{t\geq 0} X_t \geq \varepsilon\big\} \leq C(\varepsilon_1/\varepsilon)^{1/4} \quad \text{and} \quad \mathbf{P}\{\sigma_\varepsilon^Y < \infty\} \leq C(\varepsilon_1/\varepsilon)^{1/4}$$

for some constant $C > 0$, which implies

$$\mathbf{P}\{\sigma_\varepsilon^X \wedge \sigma_\varepsilon^Y < \infty\} \leq \mathbf{P}\{\sigma_\varepsilon^X < \infty\} + \mathbf{P}\{\sigma_\varepsilon^Y < \infty\} \leq 2C(\varepsilon_1/\varepsilon)^{1/4}.$$

It follows from Lemma 3.13 that

$$\mathbf{P}\{\tau_0^Y < \infty\} \geq \mathbf{P}\{\tau_0^Y \wedge \sigma_\varepsilon^X \wedge \sigma_\varepsilon^Y < \infty\} - \mathbf{P}\{\sigma_\varepsilon^X \wedge \sigma_\varepsilon^Y < \infty\} \geq 1 - 2C(\varepsilon_1/\varepsilon)^{1/4}. \tag{3.89}$$

If $X_0 > \varepsilon_1$ or $Y_0 > \varepsilon_1$, (3.89) also holds following from Lemma 3.2(i) and the Markov property. Letting $\varepsilon_1 \to 0$ we finish the proof.

**Remark 3.14.** Under Condition 1.6(i) and condition (1.10), for $q > 0$, take $\beta = p/q$ and $\varepsilon > 0$ small enough so that

$$\frac{ap}{q(q+1-\theta)} < \Big(\frac{b}{1-\theta}\Big)^{\frac{1-\theta}{q+1-\theta}} \cdot \Big(\frac{c_\theta \varepsilon^{-\theta_1}}{q}\Big)^{\frac{q}{q+1-\theta}},$$

where $\theta_1 := \beta(1-\theta) - (\kappa - p) > 0$. Then by an argument similar to that for (3.84), we get

$$-G_0(x, y) \geq x^p[-\beta a + bu^q + c_\theta \varepsilon^{-\theta_1} u^{\theta-1}], \quad 0 < x, y < \varepsilon.$$

By essentially the same argument after (3.84) in the proof of Lemma 3.13, we can also obtain (3.79) for $q > 0$ and condition (1.10). Therefore, the method for the proof of Theorem 1.9 also works for Theorem 1.7 except the case $q = 0$.

### 3.7. *Proof of Theorem 1.10*

Recall the function $G_\delta$ in (3.73). We first prove Theorem 1.10 for small related initial values of $X_0$ and $Y_0$, where the key idea, inspired by the proof of Lemma 3.13, is to consider ratio $Y_t/X_t^\beta$ process with the value of $\beta > 0$ properly selected using the conditions of Theorem 1.10. We then formulate an exponential martingale that is similar to those in [20], and use the martingale to obtain the desired estimate.

**Lemma 3.15.** *Under the conditions of Theorem 1.10, there exist constants $\beta > 0$ and small $\varepsilon > 0$ so that for $X_0 \leq \varepsilon$, $Y_0 = \varepsilon^\beta$, we have* $\mathbf{P}\{\tau_0^Y = \infty\} > 0$.

**Proof.** First note that for any $\varepsilon_2 > 0$ and $\varepsilon = \varepsilon_2^{1+\beta^{-1}}$ for any $\beta > 0$, if $X_0 \leq \varepsilon$, $Y_0 = \varepsilon^\beta$, then by Lemma 3.3 we have

$$\mathbf{P}\Big\{\sup_{s\geq 0} X_s \geq \varepsilon_2\Big\} + \mathbf{P}\Big\{\sup_{s\geq 0} Y_s \geq \varepsilon_2\Big\} \leq C[(\varepsilon^\beta \varepsilon_2^{-1})^{1/4} + (\varepsilon\varepsilon_2^{-1})^{1/4}] \leq C[\varepsilon_2^{\beta/4} + \varepsilon_2^{1/(4\beta)}], \tag{3.90}$$

where $C > 0$ is a constant independent of $\varepsilon_2$.

We first show that under conditions (i) or (ii) of Theorem 1.10, there are constants $0 < \omega < 1$, $\beta > 0$ and small enough $\delta \in (0, 1)$, $0 < \varepsilon_2 < c^*$ so that

$$G_{-\delta}(x, y) \geq 0, \qquad 0 < x, y \leq \varepsilon_2,\ x^{-\beta} y \geq \omega \tag{3.91}$$

and

$$1 - \omega^\delta - C[\varepsilon_2^{\beta/4} + \varepsilon_2^{1/(4\beta)}] > 0, \tag{3.92}$$

where $C > 0$ is the constant determined by (3.90) and $G_{-\delta}(x, y)$ is defined in (3.73).

Under condition (i) of Theorem 1.10 and Condition 1.6(ii), select $\beta$ satisfying $b/a < \beta < \kappa/(1-\theta)$. There are constants $\omega \in (0, 1)$ and small enough $\delta > 0$, $0 < \varepsilon_2 < c^*$ so that (3.92) holds and

$$(1-\beta\delta)[\beta a - b - c_\theta \omega^{\theta-1}\varepsilon_2^{\kappa-\beta(1-\theta)}] - (\beta+1)\delta b - \beta\delta c_\theta \omega^{\theta-1}\varepsilon_2^{\kappa-\beta(1-\theta)} > 0. \tag{3.93}$$

Recall function $G_0$ defined in (3.76). Under condition (i) of Theorem 1.10, $p = q = 0$ and then by Condition 1.6(ii), for all $0 < x, y \leq \varepsilon_2$ and $x^{-\beta} y \geq \omega$ we have

$$G_0(x, y) \geq \beta a - b - c_\theta (x^{-\beta} y)^{\theta-1} x^{\kappa-\beta(1-\theta)} \geq \beta a - b - c_\theta \omega^{\theta-1}\varepsilon_2^{\kappa-\beta(1-\theta)},\ G_{2,0}(y) \leq b \tag{3.94}$$

and

$$\kappa(x)\theta(y)y^{-1} \leq c_\theta x^\kappa y^{\theta-1} = c_\theta (x^{-\beta} y)^{\theta-1} x^{\kappa-\beta(1-\theta)} \leq c_\theta \omega^{\theta-1}\varepsilon_2^{\kappa-\beta(1-\theta)}. \tag{3.95}$$

Moreover, by (3.78) in Lemma 3.12, (3.93)–(3.95), for all $0 < x, y \leq \varepsilon_2$ and $x^{-\beta} y \geq \omega$, we have

$$\begin{aligned} G_{-\delta}(x, y) &\geq (1-\beta\delta)G_0(x, y) - (\beta+1)\delta G_{2,0}(y) - \beta\delta\kappa(x)\theta(y)y^{-1} \\ &\geq (1-\beta\delta)[\beta a - b - c_\theta \omega^{\theta-1}\varepsilon_2^{\kappa-\beta(1-\theta)}] - (\beta+1)\delta b - \beta\delta c_\theta \omega^{\theta-1}\varepsilon_2^{\kappa-\beta(1-\theta)} > 0, \end{aligned}$$

which gives (3.91).

Under condition (ii) of Theorem 1.10 and Condition 1.6(ii), selecting $0 < \beta < \kappa/(1-\theta)$, $\omega \in (0, 1)$ and small enough $\delta > 0$, $0 < \varepsilon_2 < c^*$ so that (3.92) holds and

$$(1-\beta\delta)[\beta a - b\varepsilon_2^q - c_\theta \omega^{\theta-1}\varepsilon_2^{\kappa-\beta(1-\theta)}] - (\beta+1)\delta b\varepsilon_2^q - \beta\delta c_\theta \omega^{\theta-1}\varepsilon_2^{\kappa-\beta(1-\theta)} > 0. \quad (3.96)$$

Since $p = 0$, then under Condition 1.6(ii), similar to (3.94), for all $0 < x, y \le \varepsilon_2$ and $x^{-\beta} y \ge \omega$ we have

$$G_0(x, y) \ge \beta a - b\varepsilon_2^q - c_\theta \omega^{\theta-1}\varepsilon_2^{\kappa-\beta(1-\theta)}, \quad G_{2,0}(y) \le b\varepsilon_2^q. \quad (3.97)$$

Thus by (3.78) in Lemma 3.12, (3.95)–(3.97) and Condition 1.6(ii) again, for all $0 < x, y \le \varepsilon_2$ and $x^{-\beta} y \ge \omega$, we have

$$\begin{aligned} G_{-\delta}(x, y) &\ge (1-\beta\delta)G_0(x, y) - (\beta+1)\delta G_{2,0}(y) - \beta\delta\kappa(x)\theta(y)y^{-1} \\ &\ge (1-\beta\delta)[\beta a - b\varepsilon_2^q - c_\theta \omega^{\theta-1}\varepsilon_2^{\kappa-\beta(1-\theta)}] - (\beta+1)\delta b\varepsilon_2^q - \beta\delta c_\theta \omega^{\theta-1}\varepsilon_2^{\kappa-\beta(1-\theta)} > 0, \end{aligned}$$

which proves (3.91).

For $v > 1$ and $\omega > 0$, define stopping times

$$\tau_w := \inf\{t \ge 0 : X_t^{-\beta} Y_t < w\} \quad \text{and} \quad \sigma_v := \inf\{t \ge 0 : X_t^{-\beta} Y_t > v\},$$

respectively. In the following we show that $\mathbf{P}\{\tau_w = \infty\} > 0$, which implies the assertion of the lemma.

Let $\omega, \beta, \delta, \varepsilon > 0$ be the constants determined in (3.91)–(3.92). Let $T := \tau_w \wedge \sigma_v$. By (1.1) and Itô's formula, for each $\delta > 0$,

$$(X_{t\wedge T}^{-\beta} Y_{t\wedge T})^{-\delta} \exp\Big\{\delta \int_0^{t\wedge T} G_{-\delta}(X_s, Y_s) \mathrm{d}s\Big\}$$

is a martingale. It follows from Fatou's lemma that

$$\begin{aligned} 1 \ge \mathbf{E}[(X_0^{-\beta} Y_0)^{-\delta}] &= \liminf_{t\to\infty} \mathbf{E}\Big[(X_{t\wedge T}^{-\beta} Y_{t\wedge T})^{-\delta} \exp\Big\{\delta \int_0^{t\wedge T} G_{-\delta}(X_s, Y_s) \mathrm{d}s\Big\}\Big] \\ &\ge \mathbf{E}\Big[\liminf_{t\to\infty} (X_{t\wedge T}^{-\beta} Y_{t\wedge T})^{-\delta} \exp\Big\{\delta \int_0^{t\wedge T} G_{-\delta}(X_s, Y_s) \mathrm{d}s\Big\}\Big] \\ &= \mathbf{E}\Big[(X_T^{-\beta} Y_T)^{-\delta} \exp\Big\{\delta \int_0^{T} G_{-\delta}(X_s, Y_s) \mathrm{d}s\Big\}\Big]. \end{aligned}$$

By (3.91) and (3.92), for $s < \tau_w$ and $\sup_{s\ge0}(X_s \vee Y_s) \le \varepsilon_2$, we have $G_{-\delta}(X_s, Y_s) \ge 0$. Then

$$\begin{aligned} 1 &\ge \mathbf{E}\Big[(X_T^{-\beta} Y_T)^{-\delta} \exp\Big\{\delta \int_0^{T} G_{-\delta}(X_s, Y_s) \mathrm{d}s\Big\} 1_{\{\sup_{s\ge0}(X_s\vee Y_s)\le\varepsilon_2, \tau_w<\sigma_v\}}\Big] \\ &\ge w^{-\delta} \mathbf{P}\Big\{\sup_{s\ge0}(X_s \vee Y_s) \le \varepsilon_2,\ \tau_w < \sigma_v\Big\}. \end{aligned}$$

By Lemma 3.2(i), we have $\lim_{v\to\infty} \sigma_v = \infty$ almost surely. Letting $v \to \infty$ in the above inequality we obtain

$$\mathbf{P}\Big\{\sup_{s\ge0}(X_s \vee Y_s) \le \varepsilon_2,\ \tau_w < \infty\Big\} \le w^{\delta}.$$

Combining (3.90) and (3.92) it follows that

$$\begin{aligned}
&\mathbf{P}\{\tau_w = \infty\} = 1 - \mathbf{P}\{\tau_w < \infty\} \\
&\geq 1 - \mathbf{P}\Big\{\sup_{s\geq 0}(X_s \vee Y_s) \leq \varepsilon_2,\ \tau_w < \infty\Big\} - \mathbf{P}\Big\{\sup_{s\geq 0}(X_s \vee Y_s) \geq \varepsilon_2\Big\} \\
&\geq 1 - w^\delta - C[\varepsilon_2^{\beta/4} + \varepsilon_2^{1/(4\beta)}] > 0,
\end{aligned}$$

which implies $\mathbf{P}\{\tau_0^Y = \infty\} > 0$ and ends the proof. □

**Proof of Theorem 1.10.** By Lemma 3.11, without loss of generality we assume that $X_0 < \varepsilon$ and $\mathbf{P}\{\sup_{t\geq 0} X_t \leq \varepsilon\} > 0$. Let $(\bar{Y}_t)_{t\geq 0}$ be the solution to (3.69) with $C_n$ replaced by $C_\varepsilon := \sup_{x\in[0,\varepsilon]} \kappa(x)$. By the comparison theorem (Proposition 3.6), we have

$$\mathbf{P}\Big\{Y_t \geq \bar{Y}_t \ \text{ for all } \ t \geq 0 \Big| \sup_{t\geq 0} X_t \leq \varepsilon\Big\} = 1.$$

Since $\mathbf{P}\{\sigma^{\bar{Y}}(\varepsilon^\beta) < \infty\} > 0$ by [20, Proposition 2.11] and $\bar{Y}$ is independent of $X$, then

$$\begin{aligned}
&\mathbf{P}\big\{X(\sigma^{\bar{Y}}(\varepsilon^\beta)) \leq \varepsilon, Y(\sigma^{\bar{Y}}(\varepsilon^\beta)) \geq \varepsilon^\beta, \sigma^{\bar{Y}}(\varepsilon^\beta) < \infty\big\} \\
&\geq \mathbf{P}\big\{X(\sigma^{\bar{Y}}(\varepsilon^\beta)) \leq \varepsilon, \bar{Y}(\sigma^{\bar{Y}}(\varepsilon^\beta)) \geq \varepsilon^\beta, \sigma^{\bar{Y}}(\varepsilon^\beta) < \infty, \sup_{t\geq 0} X_t \leq \varepsilon\big\} \\
&= \mathbf{P}\big\{\sup_{t\geq 0} X_t \leq \varepsilon, \sigma^{\bar{Y}}(\varepsilon^\beta) < \infty\big\} = \mathbf{P}\big\{\sup_{t\geq 0} X_t \leq \varepsilon\big\}\mathbf{P}\big\{\sigma^{\bar{Y}}(\varepsilon^\beta) < \infty\big\} > 0.
\end{aligned}$$

Note that by Proposition 3.6 and Lemma 3.15, there exist constants $\beta > 0$ and small $\varepsilon > 0$ so that $\mathbf{P}\{\tau_0^Y = \infty\} > 0$ if $X_0 \leq \varepsilon, Y_0 \geq \varepsilon^\beta$. Applying the strong Markov property to process $(X, Y)$ at time $\sigma^{\bar{Y}}(\varepsilon^\beta)$, we have $\mathbf{P}\{\tau_0^Y = \infty\} > 0$ for any $Y_0 > 0$, and the proof is completed. □

**Remark 3.16.** Under Condition 1.6(ii) and (iii), for $p > 0$, the corresponding estimate of $G_0(x, y)$ in Step 1 of the proof of Lemma 3.15 is not easy to establish and thus the approach of showing Theorem 1.10 does not appear to be valid for the proofs of Theorem 1.8 and Conjecture 1.11.

## Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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## Appendix

In this section we present a proof of the comparison theorem in Proposition 3.6 and show in Lemma A.1 that if all the functions in (1.1) are locally Lipschitz, then (1.1) has a pathwise unique solution. Consequently, $(X, Y)$ is a Markov process.

**Proof of Proposition 3.6.** For $k \geq 1$ define

$$h_k := \exp\{-k(k+1)/2\}.$$

Let $\psi_k$ be a nonnegative function on $\mathbb{R}$ with support in $(h_k, h_{k-1})$, $\int_{h_k}^{h_{k-1}} \psi_k(x)\mathrm{d}x = 1$ and

$$0 \leq \psi_k(x) \leq 2k^{-1}x^{-1}1_{(h_k,h_{k-1})}(x).$$

For $x \in \mathbb{R}$ and $k \geq 1$ let

$$\phi_k(x) := 1_{\{x>0\}} \int_0^x \mathrm{d}y \int_0^y \psi_k(z)\mathrm{d}z.$$

For $k \geq 1$ and $y, z \in \mathbb{R}$ put

$$\mathcal{D}_k(y,z) := \phi_k(y+z) - \phi_k(y) - z\phi_k'(y). \tag{A.1}$$

For $t \geq 0$ let $\bar{x}(t) = x_1(t) - x_2(t)$, $\hat{B}(t) = B_1(t, x_2(t)) - B_2(t, x_2(t))$, $\bar{B}(t) = B_1(t, x_1(t)) - B_1(t, x_2(t))$, $\bar{U}(t) = U(x_1(t)) - U(x_2(t))$ and $\bar{V}(t) = V(x_1(t)) - V(x_2(t))$. It then follows from (3.46) that

$$\begin{aligned}\bar{x}(t\wedge\tilde{\gamma}_n) = \bar{x}(0) &+ \int_0^{t\wedge\tilde{\gamma}_n}[\hat{B}(s)+\bar{B}(s)]\mathrm{d}s + \int_0^{t\wedge\tilde{\gamma}_n}\bar{U}(s)\mathrm{d}W_s\\ &+ \int_0^{t\wedge\tilde{\gamma}_n}\int_0^\infty\int_0^\infty g(s-,u)z\tilde{N}(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u),\end{aligned}$$

where $\tilde{\gamma}_n$ is defined in (3.47) and $g(s,u) := 1_{\{u\leq V(x_1(s))\}} - 1_{\{u\leq V(x_2(s))\}}$. Using Itô's formula we obtain

$$\begin{aligned}&\phi_k(\bar{x}(t\wedge\tilde{\gamma}_n))\\ &= \phi_k(\bar{x}(0)) + \int_0^{t\wedge\tilde{\gamma}_n}\phi_k'(\bar{x}(s))[\hat{B}(s)+\bar{B}(s)]\mathrm{d}s + \frac{1}{2}\int_0^{t\wedge\tilde{\gamma}_n}\phi_k''(\bar{x}(s))\bar{U}(s)^2\mathrm{d}s\\ &\quad+\int_0^{t\wedge\tilde{\gamma}_n}\mathrm{d}s\int_0^\infty\bar{V}(s)\mathcal{D}_k(\bar{x}(s),\mathrm{sgn}(\bar{V}(s))z)\nu(\mathrm{d}z) + \text{mart.},\end{aligned}$$

where $\mathrm{sgn}(x) = 1_{\{x>0\}} - 1_{\{x<0\}}$. It follows that

$$\begin{aligned}&\mathbf{E}\big[\phi_k(\bar{x}(t\wedge\tilde{\gamma}_n))\big]\\ &= \phi_k(\bar{x}(0)) + \mathbf{E}\Big[\int_0^{t\wedge\tilde{\gamma}_n}\phi_k'(\bar{x}(s))\hat{B}(s)\mathrm{d}s\Big]\\ &\quad+\mathbf{E}\Big[\int_0^{t\wedge\tilde{\gamma}_n}\phi_k'(\bar{x}(s))\bar{B}(s)\mathrm{d}s\Big] + \frac{1}{2}\mathbf{E}\Big[\int_0^{t\wedge\tilde{\gamma}_n}\phi_k''(\bar{x}(s))\bar{U}(s)^2\mathrm{d}s\Big]\\ &\quad+\mathbf{E}\Big[\int_0^{t\wedge\tilde{\gamma}_n}\mathrm{d}s\int_0^\infty\bar{V}(s)\mathcal{D}_k(\bar{x}(s),\mathrm{sgn}(\bar{V}(s))z)\nu(\mathrm{d}z)\Big]\\ &=: \phi_k(\bar{x}(0)) + \sum_{i=1}^4 I_{n,k}^i(t).\end{aligned} \tag{A.2}$$

For $x \in \mathbb{R}$, $x^+ := x \vee 0$. By [25, Lemma 2.1],

$$\lim_{k\to\infty}\phi_k(x) = x^+, \quad \lim_{k\to\infty}\phi_k'(x) = 1_{\{x>0\}}, \quad |x|\phi_k''(x) \leq 2k^{-1},$$

and

$$\mathcal{D}_k(y,z) \leq (2k^{-1}z^2/y)\wedge(2|z|) \quad \text{for all } k\geq 1, x, y\in\mathbb{R} \text{ and } z\geq 0 \text{ with } y(y+z)>0.$$

Then using the assumptions and the dominate convergence,

$$\lim_{k\to\infty} \mathbf{E}\big[\phi_k(\bar{x}(t\wedge\tilde{\gamma}_n))\big] = \mathbf{E}\big[\bar{x}^+(t\wedge\tilde{\gamma}_n)\big], \quad \lim_{k\to\infty} I^1_{n,k}(t) = \mathbf{E}\Big[\int_0^{t\wedge\tilde{\gamma}_n} 1_{\{\bar{x}(s)>0\}}\hat{B}(s)\mathrm{d}s\Big] \le 0$$

and

$$\lim_{k\to\infty} I^2_{n,k}(t) \le C_n\mathbf{E}\Big[\int_0^{t\wedge\tilde{\gamma}_n} \bar{x}^+(s)\mathrm{d}s\Big], \quad \lim_{k\to\infty} I^3_{n,k}(t) = \lim_{k\to\infty} I^4_{n,k}(t) = 0.$$

Combining with (A.2) we get

$$\mathbf{E}\big[\bar{x}^+(t\wedge\tilde{\gamma}_n)\big] \le \bar{x}^+(0) + C_n\mathbf{E}\Big[\int_0^{t\wedge\tilde{\gamma}_n} \bar{x}^+(s)\mathrm{d}s\Big] \le C_n\int_0^t \mathbf{E}\big[\bar{x}^+(s\wedge\tilde{\gamma}_n)\big]\mathrm{d}s.$$

From Gronwall's lemma it follows that $\mathbf{E}\big[\bar{x}^+(t\wedge\tilde{\gamma}_n)\big] = 0$. Letting $n\to\infty$ we get $\bar{x}^+(t\wedge\tilde{\gamma}) = 0$ almost surely for each fixed $t > 0$. By the right continuity of $t\mapsto x_i(t)$ $(i = 1, 2)$ we conclude the proof. □

**Lemma A.1.** *Suppose that the functions $a_i, b_i, i = 1, 2, 3$ and $\theta, \kappa$ in (1.1) are all locally Lipschitz, i.e., for each $m, n \ge 1$, there is a constant $C_{m,n} > 0$ so that*

$$\sum_{i=1,2,3}\big[|a_i(x) - a_i(y)| + |b_i(x) - b_i(y)|\big] \le C_{m,n}|x - y|, \qquad x, y \in [n^{-1}, m].$$

*Then SDE (1.1) has a nonnegative pathwise unique solution.*

**Proof.** For $n \ge 1$ and $i = 1, 2, 3$ let $a_i^n(x) := a_i((x\wedge n)\vee n^{-1})$. Define $b_i^n$ and $\theta^n, \kappa^n$, similarly. Inspired by the argument in [10, Theorem 3.1], let

$$U = \{1, 2\}\times(0,\infty)^2, \quad U_0 = \big(\{1\}\times(0, 1)\times(0,\infty)\big)\cup\big(\{2\}\times(0, 1)\times(0,\infty)\big).$$

Let

$$\tilde{N}_0(\mathrm{d}s, \mathrm{d}v, \mathrm{d}z, \mathrm{d}u) := \delta_1(\mathrm{d}v)\tilde{M}(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u) + \delta_2(\mathrm{d}v)\tilde{N}(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u).$$

Then $\tilde{N}_0$ is a compensated Poisson random measure on $(0,\infty)\times U$ with intensity

$$\mathrm{d}s[\delta_1(\mathrm{d}v)\mu(\mathrm{d}z) + \delta_2(\mathrm{d}v)\nu(\mathrm{d}z)]\mathrm{d}u.$$

Define functions $f_1^n$ and $f_2^n$ on $[0,\infty)\times U$ by

$$f_1^n(x, v, z, u) := z1_{\{v=1, u\le a_3^n(x)\}}, \quad f_2^n(x, v, z, u) := z1_{\{v=2, u\le b_3^n(x)\}}.$$

Write $\mathbf{u}$ for $(v, z, u)$. Then SDE (1.1) can be written into this form

$$\begin{cases} X_t = X_0 - \int_0^t \Big[a_1^n(X_s) + a_3^n(X_s)\int_1^\infty z\mu(\mathrm{d}z)\Big]\mathrm{d}s + \int_0^t a_2^n(X_s)^{1/2}\mathrm{d}B_s \\ \qquad + \int_0^t\int_{U_0} f_1^n(X_{s-}, \mathbf{u})\tilde{N}_0(\mathrm{d}s, \mathrm{d}\mathbf{u}) + \int_0^t\int_{U\setminus U_0} f_1^n(X_{s-}, \mathbf{u})N_0(\mathrm{d}s, \mathrm{d}\mathbf{u}), \\ Y_t = Y_0 + \int_0^t \Big[-b_1^n(Y_s) - \theta^n(Y_s)\kappa^n(X_s) + b_3^n(X_s)\int_1^\infty z\nu(\mathrm{d}z)\Big]\mathrm{d}s + \int_0^t b_2^n(Y_s)^{1/2}\mathrm{d}W_s \\ \qquad + \int_0^t\int_{U_0} f_2^n(Y_{s-}, \mathbf{u})\tilde{N}_0(\mathrm{d}s, \mathrm{d}\mathbf{u}) + \int_0^t\int_{U\setminus U_0} f_2^n(Y_{s-}, \mathbf{u})N_0(\mathrm{d}s, \mathrm{d}\mathbf{u}), \end{cases} \tag{A.3}$$

where $N_0$ is the corresponding Poisson random measure of $\tilde{N}_0$. It follows from [14, p. 245] that (A.3) has a strong unique solution $(X_t^n, Y_t^n)_{t\ge0}$. Letting $n\to\infty$, by the same argument in [20, Theorem 3.1 (i)], SDE (1.1) has a pathwise unique solution.

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