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Invariance principle for the maximal position process of branching Brownian motion in random environment*†

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Abstract

In this paper we study the maximal position process of branching Brownian motion in random spatial environment. The random environment is given by a process $\xi = (\xi(x))_{x\in\mathbb{R}}$ satisfying certain conditions. We show that the maximum position M_t of particles alive at time t satisfies a quenched strong law of large numbers and an annealed invariance principle.

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1 Introduction

1.1 Background and assumptions

A binary branching Brownian motion (BBM) in $\mathbb{R}$ can be described as follows: There is an initial particle starting from the origin, and the particle moves as a standard Brownian motion. After an independent exponential amount of time with parameter one, the particle dies and splits into two new particles. The offspring evolve independently as their parent.

Let M_t denote the maximal position among all the particles alive at time t. Kolmogorov, Petrovskii and Piskounov [15] showed that $\frac{M_t}{t}$ converges to $\sqrt{2}$ in probability as $t \to \infty$.

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In [3, 4], Bramson proved that, as $t \to \infty$, $M_t - (\sqrt{2}t - \frac{3}{2\sqrt{2}} \ln t)$ converges weakly to a limit related to a travelling wave solution. In [17], Lalley and Sellke gave a probabilistic representation of the limit. Thus in this classical case, $M_t = \sqrt{2}t + O(\ln t)$ as $t \to \infty$.

Černý and Drewitz [7] studied continuous-time branching random walks in random environment and their results show that the asymptotic behavior of branching random walk in random environment is different from the classical case. A continuous-time branching random walk in random environment can be described as follows: Suppose that $\xi = \{\xi(x) : x \in \mathbb{Z}\}$ is a family of iid random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with $0 < \text{essinf}\, \xi(0) < \text{esssup}\, \xi(0) < \infty$. Let

$$\Omega_0 := \{\omega : \{\xi(x, \omega), x \in \mathbb{Z}\} \text{ is a sequence of positive numbers}\}.$$

Then $\mathbb{P}(\Omega_0) = 1$. For any $\omega \in \Omega_0$ and $x \in \mathbb{Z}$, let P_x^ξ denote the law of a continuous-time binary branching random walk starting from x with branching rate $\xi(\cdot, \omega)$. There is a particle at x at time 0. As time evolves, the particle moves as a continuous time random walk with jump rate 1. In addition, and independently of everything else, while at site y, a particle splits into two at rate $\xi(y, \omega)$, and when it does so, the two new particles evolve independently according to the same diffusion mechanism as their parent. P_x^ξ is referred to as the quenched law and $\mathbb{P} \times \mathrm{P}_x^\xi$ is referred to as the annealed law.

We still use M_t to denote the maximal displacement at time t of branching random walk in random environment. Under some conditions, Černý and Drewitz [7] proved an annealed invariance principle for M_t. That is, there exist $v_0 > 0$ and $\sigma_0 > 0$ such that, under $\mathbb{P} \times \mathrm{P}_0^\xi$, the sequence of processes

$$[0, \infty) \ni t \mapsto \frac{M_{nt} - v_0 nt}{\sigma_0 \sqrt{n}}, \quad n \in \mathbb{N},$$

converges, as $n \to \infty$, weakly to a standard Brownian motion. Thus $\mathbb{P}$-almost surely, $M_t \neq v_0 t + O(\ln t)$ as $t \to \infty$ under P_0^ξ.

In this paper, we study branching Brownian motion in random environment (BBMRE). As in [7], we suppose that $\{\xi(x), x \in \mathbb{R}\}$ is a family of non-negative random variables on $(\Omega, \mathcal{F}, \mathbb{P})$. We will add conditions on $\{\xi(x), x \in \mathbb{R}\}$ below. For any fixed $\omega \in \Omega$, $\{\xi(x, \omega), x \in \mathbb{R}\}$ can be regarded as a branching rate. Branching Brownian motion in random environment ξ started at $x \in \mathbb{R}$ can be defined as follows: Conditionally on a realization of ξ, we place one particle at x at time 0. As time evolves, all particles move independently according to a standard Brownian motion. In addition, and independently of everything else, while at y, a particle splits at rate $\xi(y)$. Once a particle splits, this particle is removed and, randomly and independently from everything else, replaced by k new particles, with probability p_k, at the position of the removed particle. These k new particles evolve independently according to the same diffusion-branching mechanism as their parent. For a given $\xi(\cdot, \omega)$, let P_x^ξ denote the law of branching Brownian motion starting from x with spatial-dependent branching rate $\xi(\cdot, \omega)$ and offspring distribution $\{p_k : k = 0, 1, 2, ...\}$. More precisely, let $N(t)$ be the set of particles alive at time t and X_t^ν be the position of the particle $\nu \in N(t)$. Define

$$X_t := \sum_{\nu \in N(t)} \delta_{X_t^\nu}, \quad t \geq 0,$$

which is the point process generated by the position of the particles alive at time t. We call $\{X_t, t \geq 0\}$ a branching Brownian motion in random in random environment ξ.

For any $\nu \in N(t)$ and $s \in [0, t]$, we use X_s^ν to denote the position of ν or its ancestor at time s. $X_\cdot^\nu$ is referred to as the genealogy of $\nu \in N(t)$.

For any $t \geq 0$, put

$$M_t := \max_{\nu \in N(t)} X_t^\nu.$$

The purpose of this paper is to study the limit behaviour of M_t as $t \to \infty$ under the quenched law P_x^ξ, $x \in \mathbb{R}$, and prove an invariance principle for M_t under the annealed law $\mathbb{P} \times \mathrm{P}_0^\xi$. Throughout this paper, we assume that $p_0 = 0$ and $\sum_{k=1}^\infty kp_k = m > 1$. When $p_0 > 0$, we can consider M_t under $\mathrm{P}_x^\xi(\cdot|\text{ survival})$ and $\mathbb{P} \times \mathrm{P}_x^\xi(\cdot|\text{ survival})$.

The first assumption **(H1)** on the environment ξ already appeared in [1, 8, 10, 20, 21].

(H1) **(a)** $\mathbb{P}$-almost surely, ξ is uniformly Hölder continuous, i.e., there exist constants $C(\xi), \alpha(\xi) > 0$ such that

$$|\xi(x,\omega) - \xi(y,\omega)| \le C(\xi)|x-y|^{\alpha(\xi)}, \quad \forall\, x, y \in \mathbb{R}.$$

(b) There exist constants $0 < \mathrm{ei} \le \mathrm{es} < \infty$ such that $\mathbb{P}$-a.s.,

$$\mathrm{ei} \le \xi(x) \le \mathrm{es}, \quad \forall\, x \in \mathbb{R}.$$

(c) There exists a group of measure-preserving transformations $\{\theta_y\}_{y\in\mathbb{R}}$, acting ergodically on $(\Omega, \mathcal{F}, \mathbb{P})$, such that, for almost every $\omega \in \Omega$, $\xi(x+y,\omega) = \xi(x, \theta_y\omega)$ holds for all $x, y \in \mathbb{R}$.

(d) ξ satisfies the ζ-mixing condition: Let $\mathcal{F}_x := \sigma\left(\xi(t) : t \le x\right)$, $\mathcal{F}^y := \sigma\left(\xi(r) : r \ge y\right)$ and $\zeta : [0,\infty) \to [0,\infty)$ be a continuous non-increasing function with $\sum_{k=0}^\infty \zeta(k) < \infty$. It holds that for all $x, y \in \mathbb{R}$ with $x \le y$ and all $X \in L^1(\Omega, \mathcal{F}_x, \mathbb{P})$ and $Y \in L^1(\Omega, \mathcal{F}^y, \mathbb{P})$,

$$|\mathbb{E}\left((X - \mathbb{E}(X)) \mid \mathcal{F}^y\right)| \le \mathbb{E}\left(|X|\right) \cdot \zeta(y-x),$$
$$|\mathbb{E}\left((Y - \mathbb{E}(Y)) \mid \mathcal{F}_x\right)| \le \mathbb{E}\left(|Y|\right) \cdot \zeta(y-x).$$

The mixing condition above is given in [10], and is weaker than the one given in [21]. For simplicity, we assume that **(c)** holds for all $\omega \in \Omega$ rather than $\mathbb{P}$-almost surely.

Our second assumption is [10, (VEL)], but we will describe it using principal eigenvalues, see Remark 1.1 below. Define

$$\mathcal{A}_\infty := \left\{\phi \in \mathcal{C}^2(\mathbb{R}), \frac{\phi_x}{\phi} \in L^\infty(\mathbb{R}), \phi > 0 \text{ in } \mathbb{R}, \lim_{|x|\to+\infty} \frac{1}{x}\ln\phi(x) = 0\right\}.$$

For $\lambda \in \mathbb{R}$ and $\phi(x) \in \mathcal{C}^2(\mathbb{R})$, define

$$L_\lambda^\omega \phi := \frac{1}{2}\phi_{xx} - \lambda\phi_x + \left(\frac{\lambda^2}{2} + (m-1)\xi(x,\omega)\right)\phi.$$

For any $\lambda \in \mathbb{R}$ and non-empty open interval $I \subset \mathbb{R}$, define

$$\begin{aligned}
\underline{\gamma}(L_\lambda^\omega) &:= \sup\left\{\gamma \middle| \;\exists\, \phi \in \mathcal{A}_\infty \text{ such that } L_\lambda^\omega\phi \ge \gamma\phi \text{ in } \mathbb{R}\right\},\\
\overline{\gamma}(L_\lambda^\omega) &:= \inf\left\{\gamma \middle| \;\exists\, \phi \in \mathcal{A}_\infty \text{ such that } L_\lambda^\omega\phi \le \gamma\phi \text{ in } \mathbb{R}\right\},\\
\Lambda_1(L_0^\omega, I) &:= \inf\left\{\gamma \middle| \;\exists \phi \in \mathcal{C}^2(I)\cap\mathcal{C}^0(\overline{I}),\ \phi > 0 \text{ in } I,\ \phi = 0 \text{ in } \partial I,\ L_0^\omega\phi \le \gamma\phi \text{ in } I\right\},\\
\Lambda_1(L_0^\omega) &:= \lim_{R\to+\infty} \Lambda_1\left(L_0^\omega, (-R,+R)\right).
\end{aligned}$$

The definitions of $\underline{\gamma}(L_\lambda^\omega)$ and $\overline{\gamma}(L_\lambda^\omega)$ are special cases of Berestycki and Nadin [1, (10) and (11)] with $R = -\infty$, and the definitions of $\Lambda_1(L_0^\omega, I)$ and $\Lambda_1(L_0^\omega)$ are given in [1, (45) and (47)]. By [1, Theorem 5.1 (1)], there exists $\Omega_1 \subset \Omega$ with $\mathbb{P}(\Omega_1) = 1$ such that, for any $\omega \in \Omega_1$ and $\lambda \in \mathbb{R}$, $\underline{\gamma}(L_\lambda^\omega) = \overline{\gamma}(L_\lambda^\omega)$ and this common value does not depend on $\omega \in \Omega_1$. We will write $\underline{\gamma}(L_\lambda^\omega)$ as $\gamma(\lambda)$ for simplicity. By [1, (51) and Lemma 5.1], we know that there exist two constants $\rho_L \le 0$ and $\rho_R \ge 0$ such that when $\lambda \in [\rho_L, \rho_R]$, $\gamma(\lambda) = \Lambda_1(L_0^\omega)$. Nadin [20, Lemma 3.2] proved that

$$\rho_R = -\rho_L =: \rho \ge 0.$$

From the argument of [1, Theorem 5.1], we know that when $\lambda \notin [\rho_L, \rho_R]$, $\gamma(\lambda) > \Lambda_1(L_0^\omega)$, and in this case, by [1, Theorem 5.1 (2)], there exists $\phi = \phi(\cdot, \lambda, \omega) \in \mathcal{A}_\infty$ satisfying

$$L_\lambda^\omega \phi = \frac{1}{2}\phi_{xx} - \lambda\phi_x + \left(\frac{\lambda^2}{2} + (m-1)\xi(x,\omega)\right)\phi = \gamma(\lambda)\phi. \tag{1.1}$$

Moreover, the proof of [1, Theorem 5.1 (1)] shows that $\gamma(\cdot)$ is positive and strictly increasing in $(\rho, +\infty)$. Also, Nadin [20, Lemma 3.2] shows that $\gamma(\cdot)$ is even. By [20, Lemma 3.3], ϕ is unique up to multiplicative constant. Thus, we can assume that $\phi(0,\lambda) = 1$ for all $|\lambda| > \rho$ and $\omega \in \Omega_1$ without loss of generality. Also [20, Proposition 1 and Proposition 3] imply that for all $\lambda \in \mathbb{R}$,

$$(m-1)\mathrm{es} + \frac{\lambda^2}{2} \geq \gamma(\lambda) \geq (m-1)\mathrm{ei} + \frac{\lambda^2}{2},$$

so v^* and λ^* below are well-defined:

$$v^* := \min_{\lambda>0} \frac{\gamma(\lambda)}{\lambda}, \quad \lambda^* := \operatorname*{argmin}_{\lambda>0} \frac{\gamma(\lambda)}{\lambda}.$$

Our second assumption is as follows:

(H2) $\gamma(\lambda^*) > (m-1)\mathrm{es}$.

Remark 1.1. Here we explain why **(H2)** corresponds to [10, (VEL)]. The function $L(\eta)$ defined in [10, (2.8)] is strictly increasing for $\eta < 0$. By [10, Lemma 2.4 (a) and (d)], $L'(\eta)$ is strictly increasing for $\eta < 0$ and, for every $v > v_c := \frac{1}{L'(0-)}$, there exists a unique non-random $\overline{\eta}(v) < 0$ such that $vL'(\overline{\eta}(v)) = 1$. So the condition [10, (VEL)] "$v_0 > v_c$" is equivalent to $\overline{\eta}(v_0) < \overline{\eta}(v_c) = 0$, which is also equivalent to $L(\overline{\eta}(v_0)) < L(0)$ by the monotonicity of L. Combining [6, (4.10)] with (1.6) of this paper (see below), one sees that there is a one-to-one correspondence between our notation $\gamma(\lambda)$ and η in [10]: $\gamma(\lambda) = -\eta + (m-1)\mathrm{es}$. Therefore, we can translate the condition $L(\overline{\eta}(v_0)) < L(0)$ into **(H2)**.

Since $\gamma(\rho) = \gamma(0) \leq (m-1)\mathrm{es}$, **(H2)** implies $\lambda^* > \rho$. Combining [10, Proposition 4.10] with the fact that **(H2)** and [10, (VEL)] are equivalent, we know that there exists a ξ satisfying **(H1)** but not **(H2)**.

Our final assumption **(H3)** will be used when we deal with the invariance principle for M_t.

(H3) $m_2 := \sum_{k=1}^\infty k^2 p_k < +\infty$.

BBMRE is related to the random F-KPP:

$$w_t = \frac{1}{2}w_{xx} + \xi(x,\omega)\left(1 - w - \sum_{k=1}^\infty p_k(1-w)^k\right). \tag{1.2}$$

Solutions to (1.2) can be written as

$$w(t,x) = 1 - \mathrm{E}_x^\xi\left(\prod_{\nu\in N(t)} (1 - w(0, X_t^\nu))\right), \tag{1.3}$$

see [10, Proposition 2.1]. (1.3) is referred to as McKean's representation, see [19] for the case of homogeneous branching Brownian motion. In particular, $w(t,x) = \mathrm{P}_x^\xi(M_t \geq 0)$ solves (1.2) with $w(0,x) = 1_{[0,\infty)}(x)$, and $w(t,x) = \mathrm{P}_x^\xi\left(\min_{\nu\in N(t)} X_t^\nu \leq 0\right)$ solves (1.2) with $w(0,x) = 1_{(-\infty,0]}(x)$.

Freidlin [12, Chapter 7] studied branching Brownian motion in random environment by considering (1.2). The asymptotic wave front propagation velocity for (1.2) is of special interest. By [12, Theorem 7.6.1], under suitable assumptions, the solution $w(t,x)$ to (1.2) converges to 0 (resp. 1), uniformly for all $x \geq vt$ with $v > v^*$ (resp. for all $x \leq vt$ with $v \in (0, v^*)$), as $t \to \infty$. In particular, using [12, Theorem 7.6.1] with $w(t,x) = \mathrm{P}_x^\xi \left(\min_{\nu \in N(t)} X_t^\nu \leq 0 \right)$ (which implies $w(0,x) = 1_{(-\infty,0]}(x)$), we have that $\mathbb{P}$-a.s. $\lim_{t\to\infty} \frac{m(t)}{t} = v^*$, where

$$m(t) := \sup \left\{ x : \mathrm{P}_x^\xi \left(\min_{\nu \in N(t)} X_t^\nu \leq 0 \right) = \frac{1}{2} \right\}$$

is known as the front of (1.2). In particular when $\xi(x,\omega) \equiv c > 0$ with c being a positive constant, $v^* = \sqrt{2c(m-1)}$. This result is due to Kolmogorov-Petrovskii-Piskunov [15].

Nolen [21] proved a central limit theorem for $\mathcal{X}_t := \sup\{x : w(t,x) = \frac{1}{2}\}$, where the assumption on the initial value $w(0,x)$ of (1.2) falls into the supercritical regime — the limit of $\frac{\mathcal{X}_t}{t}$ is larger than the minimal speed v^*. Recently, Drewitz and Schmitz [10] studied the case when $w(0,x)$ satisfies

$$0 \leq w(0,x) \leq 1_{(-\infty,0]}(x), \quad \text{and} \int_{[x-N,x]} w(0,y)dy \geq \delta, \quad \forall x \leq -N',$$

for some fixed $\delta, N, N' > 0$. This case corresponds to the critical regime. [10] proved an invariance principle for $m^\epsilon(t) := \sup \left\{ x \in \mathbb{R} : \mathrm{P}_x^\xi \left(\min_{\nu \in N(t)} X_t^\nu \leq 0 \right) \geq \epsilon \right\}$ with $\epsilon \in (0,1)$. Note that

$$\mathrm{P}_x^\xi \left(\min_{\nu \in N(t)} X_t^\nu \leq 0 \right) = \mathrm{P}_x^\xi \left(\max_{\nu \in N(t)} (-X_t^\nu) \geq 0 \right),$$

and that $\{\sum_{\nu \in N(t)} \delta_{-X_t^\nu}, t \geq 0\}$ is a BBMRE starting at $-x$ with branching rate $\tilde{\xi}(x) = \xi(-x)$. Therefore,

$$m^\epsilon(t) = \sup\{x \in \mathbb{R} : \mathrm{P}_{-x}^{\tilde{\xi}}(M_t \geq 0) \geq \epsilon\}.$$

By the translation and reflection invariance of the environment ξ, we have for $x \in \mathbb{R}$,

$$\mathrm{P}_{-x}^\xi(M_t \geq 0) = \mathrm{P}_0^\xi(M_t \geq x) \quad \text{in distribution under } \mathbb{P}.$$

This says that the invariance principle for $m^\epsilon(t)$ in [10] is related to our invariance principle for M_t. In the homogeneous medium, i.e., $\xi(x,\omega) \equiv c$, $\mathrm{P}_x^c \left(\min_{\nu \in N(t)} X_t^\nu \leq 0 \right) = \mathrm{P}_0^c \left(M_t \geq x \right)$, and then $m^\epsilon(t)$, the front of the solution to (1.2) coincides with the ϵ median of the distribution of the maximal particle of the BBM. In the random medium case, $m^\epsilon(t)$ has the same distribution as the ϵ median of the distribution of the maximal particle of BBMRE. In this paper, we are mainly interested in the behavior of $M_t, t \geq 0$.

We will always use $\{(B_t)_{t\geq 0}; \Pi_x\}$ to denote a standard Brownian motion starting from x at $t = 0$, and also use Π_x for expectation with respect to Π_x for simplicity. According to [10, Proposition 2.3], we have the following many-to-one and many-to-two formulae:

Proposition 1.2. *Let $\varphi_1, \varphi_2 : [0,\infty) \to [-\infty, \infty]$ be càdlàg functions with $\varphi_1 \leq \varphi_2$. Then the first and second moments of the number of particles in $N(t)$ with genealogy staying between φ_1 and φ_2 in the time interval $[0,t]$ are given by*

$$\begin{aligned} &\mathrm{E}_x^\xi \left(\# \left\{ \nu \in N(t) : \varphi_1(s) \leq X_s^\nu \leq \varphi_2(s),\ \forall s \in [0,t] \right\} \right) \\ &\quad = \Pi_x \left(\exp\left\{ \int_0^t (m-1)\xi\left(B_r\right) \mathrm{d}r \right\}; \varphi_1(s) \leq B_s \leq \varphi_2(s),\ \forall s \in [0,t] \right) \end{aligned}$$

and

$$\begin{aligned}
&\mathrm{E}_x^\xi\left(\#\left\{\nu\in N(t):\varphi_1(s)\le X_s^\nu\le\varphi_2(s)\ \forall s\in[0,t]\right\}^2\right)\\
&=\Pi_x\left(\exp\left\{\int_0^t(m-1)\xi\left(B_r\right)\mathrm{d}r\right\};\varphi_1(s)\le B_s\le\varphi_2(s)\ \forall s\in[0,t]\right)\\
&+(m_2-m)\int_0^t\Pi_x\Big(\exp\left\{\int_0^s(m-1)\xi\left(B_r\right)\mathrm{d}r\right\}\xi\left(B_s\right)1_{\{\varphi_1(r)\le B_r\le\varphi_2(r),\ \forall 0\le r\le s\}}\\
&\quad\times\left(\Pi_y\left(\exp\left\{\int_0^{t-s}(m-1)\xi\left(B_r\right)\mathrm{d}r\right\}1_{\{\varphi_1(r+s)\le B_r\le\varphi_2(r+s)\ \forall 0\le r\le t-s\}}\right)\right)^2_{|y=B_s}\Big)\mathrm{d}s
\end{aligned}$$

respectively.

For $a\in\mathbb{R}$, letting $\varphi_1\equiv-\infty$, $\varphi_2(s)=\begin{cases}+\infty, & \text{if } s<t,\\ a, & \text{if } s=t\end{cases}$ in Proposition 1.2, we get that

$$\mathrm{E}_x^\xi\left(\sum_{\nu\in N(t)}f\left(X_t^\nu\right)\right)=\Pi_x\left(\exp\left\{\int_0^t(m-1)\xi(B_r)\mathrm{d}r\right\}f(B_t)\right),\tag{1.4}$$

first for $f(x)=1_{(-\infty,a]}(x)$, and then for any non-negative Borel function f.

Recall that for $|\lambda|>\rho$ and $\omega\in\Omega_1$, $\phi(x,\lambda,\omega)$ solves (1.1) with $\phi(0,\lambda,\omega)=1$. Define

$$\psi(x,\lambda):=e^{-\lambda x}\phi(x,\lambda),\quad x\in\mathbb{R}\tag{1.5}$$

and

$$u_{\psi(\lambda)}(t,x):=e^{\gamma(\lambda)t}\psi(x,\lambda)=e^{\gamma(\lambda)t}e^{-\lambda x}\phi(x,\lambda),\quad x\in\mathbb{R}.$$

Then $\psi(\cdot,\lambda)$ solves the problem

$$\begin{cases}\frac{1}{2}\psi_{xx}+(m-1)\xi(x,\omega)\psi=\gamma(\lambda)\psi, & x\in\mathbb{R},\\ \psi(0,\lambda,\omega)=1,\end{cases}\tag{1.6}$$

and $u_{\psi(\lambda)}$ solves the problem

$$\begin{cases}u_t=\frac{1}{2}u_{xx}+(m-1)\xi(x,\omega)u, & t>0,\quad x\in\mathbb{R},\\ u(0,x,\lambda)=\psi(x,\lambda).\end{cases}$$

By the Feynman-Kac formula, $u_{\psi(\lambda)}(t,x)$ has the following representation

$$u_{\psi(\lambda)}(t,x)=\Pi_x\left(\exp\left\{\int_0^t(m-1)\xi(B_s,\omega)\mathrm{d}s\right\}e^{-\lambda B_t}\phi(B_t,\lambda)\right).\tag{1.7}$$

Thus

$$\begin{aligned}
&e^{-\lambda x}\phi(x,\lambda)=e^{-\gamma(\lambda)t}u_{\psi(\lambda)}(t,x)\\
=&e^{-\gamma(\lambda)t}\Pi_x\left(\exp\left\{\int_0^t(m-1)\xi(B_s,\omega)\mathrm{d}s\right\}e^{-\lambda B_t}\phi(B_t,\lambda)\right).
\end{aligned}$$

By (1.4), one has

$$e^{-\lambda x}\phi(x,\lambda)=e^{-\gamma(\lambda)t}\mathrm{E}_x^\xi\left(\sum_{\nu\in N(t)}e^{-\lambda X_t^\nu}\phi(X_t^\nu,\lambda)\right).\tag{1.8}$$

Let

$$W_t(\lambda) := e^{-\gamma(\lambda)t} \sum_{\nu \in N(t)} e^{-\lambda X_t^\nu} \phi(X_t^\nu, \lambda), \quad t \geq 0,$$

and $\widetilde{\mathcal{F}}_t$ be the σ-field generated by all information of the branching Brownian motion up to time t. Using (1.8) and the Markov property of $\{X_t, t \geq 0\}$, we have the following lemma:

Lemma 1.3. *For any* $|\lambda| > \rho$ *and* $\omega \in \Omega_1$, $\left(W_t(\lambda), \widetilde{\mathcal{F}}_t, \mathrm{P}_x^\xi\right)$ *is a positive martingale with mean* $e^{-\lambda x}\phi(x, \lambda)$.

We omit the proof here, see [22, Lemma 1.2] for a proof in a more general case.

The first purpose of the present paper is to study the quenched limit of $W_t(\lambda)$ as $t \to \infty$ under Assumptions **(H1)** and **(H2)**, see Theorem 1.4. As a consequence, we will get that, when $\sum_{k=1}^\infty (k \ln k) p_k < \infty$, $\mathbb{P}$-almost surely,

$$\frac{M_t}{t} \to v^*, \ \mathrm{P}_x^\xi - \text{a.s.} \quad \text{as } t \to \infty,$$

see Corollary 1.5.

The second purpose of this paper is to prove an annealed invariance principle for M_t under Assumptions **(H1)**, **(H2)** and **(H3)**.

Conditioned on ξ, for any non-negative Borel function f on $\mathbb{R}$, $u(t,x) := u_f(t,x) = \mathrm{E}_x^\xi\left(\sum_{\nu \in N(t)} f(X_t^\nu)\right)$ solves the following parabolic Anderson problem:

$$\begin{cases} \frac{\partial}{\partial t} u(t,x) = \frac{1}{2} u_{xx} + (m-1)\xi(x,\omega) u(t,x), \quad x \in \mathbb{R}, \\ u(0,x) = f(x). \end{cases} \tag{1.9}$$

According to [21, Section 1], [7, Section 6] and [10, Section 4], the maximal position M_t of X_t is related to the front of the solution u_f. We write $u^{(-\infty,0]}$ for the solution to (1.9) with initial condition $u(0,x) = 1_{(-\infty,0]}(x)$. The front of $u^{(-\infty,0]}$ is defined as

$$\overline{m}^\epsilon(t) = \sup\{x \in \mathbb{R} : u^{(-\infty,0]}(t,x) \geq \epsilon\}.$$

[10, Theorem 1.4] proved that there exist a constant $c \in (0,\infty)$ and a $\mathbb{P}$-a.s. finite random time $T(\omega)$ such that for all $t \geq T(\omega)$,

$$\overline{m}^\epsilon(t) - m^\epsilon(t) \leq c \ln t.$$

The processes $\{\overline{m}^\epsilon(t), t \geq 0\}$ and $\{m^\epsilon(t), t \geq 0\}$ satisfy invariance principles, see [10, Theorem 1.3, Corollary 1.5].

Nolen [21] studied the front of u_f with $f = \psi(x, -\lambda^*)$. Note that

$$u_{\psi(-\lambda^*)}(t,x) = e^{\gamma(-\lambda^*)t}\psi(x, -\lambda^*, \omega) = e^{\gamma(\lambda^*)t} e^{\lambda^* x}\phi(x, -\lambda^*, \omega) \tag{1.10}$$

solves (1.9) with $f = \psi(x, -\lambda^*)$. The position of the wave $u_{\psi(-\lambda^*)}$ at time t is defined in [21] by

$$\overline{m}_\psi^\epsilon(t,\omega) := \sup\left\{x \in \mathbb{R} : u_{\psi(-\lambda^*)}(t,x,\omega) = \epsilon\right\}.$$

Using the fact that $\log(\psi(x, -\lambda^*, \omega)) \sim \lambda^* x$ as $x \to \infty$, we have

$$\lim_{t\to\infty} \frac{\overline{m}_\psi^\epsilon(t,\omega)}{t} = -\frac{\gamma(\lambda^*)}{\lambda^*} = -v^* \quad \mathbb{P}\text{-a.s.}, \tag{1.11}$$

see a similar argument in [21, (5)].

In this paper, we will introduce a process V_t to play the role of $-\overline{m}^{\epsilon}_{\psi}(t,\omega)$. **(H2)** implies that $|-\lambda^*| > \rho$. Taking $\lambda = -\lambda^*$ and $x = 0$ in (1.8) and using the fact that $\gamma(-\lambda^*) = \gamma(\lambda^*)$, we have

$$e^{\gamma(\lambda^*)t} = \mathrm{E}_0^{\xi}\left(\sum_{\nu\in N(t)} \psi(X_t^{\nu}, -\lambda^*)\right). \tag{1.12}$$

Since $\psi(x,-\lambda^*)$ grows exponentially as a function of x, we know that $\psi(M_t,-\lambda^*)$ makes the largest contribution to the sum in (1.12). Using this observation, we define V_t by

$$V_t := \sup\left\{x\in\mathbb{R} : \psi(x,-\lambda^*,\omega) = e^{\gamma(\lambda^*)t}\right\} = \sup\left\{x\in\mathbb{R} : \lambda^* x + \ln\phi(x,-\lambda^*,\omega) = \gamma(\lambda^*)t\right\}. \tag{1.13}$$

Since $\psi(\cdot,-\lambda^*,\omega)$ is continuous, the supremum above is attained. Thus, $\psi(V_t,-\lambda^*) = e^{\gamma(\lambda^*)t}$. If $\nu\in N(t)$ is such that $X_t^{\nu} = V_t$, then, since $\psi(\cdot,-\lambda^*,\omega)$ is strictly increasing (see Remark 2.4 below), $\psi(X_t^{\nu},-\lambda^*)$ makes the largest contribution to the sum of the right hand side of (1.12). By (1.10), we may rewrite our V_t as

$$V_t(\omega) = \sup\left\{x : u_{\psi(-\lambda^*)}(t,x,\omega) = e^{2\gamma(\lambda^*)t}\right\}.$$

By (1.11), we have

$$\lim_{t\to\infty}\frac{V_t}{t} = v^* \quad \mathbb{P}\text{-a.s.},$$

Although both V_t and $\overline{m}^{\epsilon}(t)$ are fronts of the linear problem (1.9) at different levels, we find that V_t is easier to handle than $\overline{m}^{\epsilon}(t)$ when we investigate the main contribution to $\sum_{\nu\in N(t)}\psi(X_t^{\nu},-\lambda^*)$.

1.2 Main results

Theorem 1.4. *If **(H1)** and **(H2)** hold, then there exists $\Omega_2\subset\Omega_1$ with $\mathbb{P}(\Omega_2)=1$ such that for any $\omega\in\Omega_2$, the limit $W_\infty(\lambda) := \lim_{t\to\infty} W_t(\lambda)$ exists P_x^{ξ}-a.s. Moreover,*

(i) if $|\lambda|\geq\lambda^*$ then $W_\infty(\lambda)=0,\ \mathrm{P}_x^{\xi}$-a.s.;

(ii) if $|\lambda|\in(\rho,\lambda^*)$, then $W_\infty(\lambda)$ is an $L^1(\mathrm{P}_x^{\xi})$ limit or $W_\infty(\lambda)=0$ according to $\sum_{k=1}^{\infty}(k\ln k)p_k<\infty$ or $\sum_{k=1}^{\infty}(k\ln k)p_k=\infty$.

Corollary 1.5. *Assume that **(H1)** and **(H2)** hold. If $\sum_{k=1}^{\infty}(k\ln k)p_k<\infty$, then $\mathbb{P}$-almost surely, $M_t/t\to v^*,\ \mathrm{P}_x^{\xi}$-a.s. as $t\to\infty$.*

Theorem 1.6. *Assume that **(H1)**, **(H2)** and **(H3)** hold. There exists a non-random constant $\Gamma>0$ such that for $\mathbb{P}$-a.s. ω,*

$$\limsup_{t\to\infty}\frac{|M_t-V_t|}{\ln t}\leq\Gamma, \qquad \mathrm{P}_0^{\xi}\text{-a.s.}.$$

Theorem 1.7. *Assume that **(H1)**, **(H2)** and **(H3)** hold.*

(i) Under $\mathbb{P}\times\mathrm{P}_0^{\xi}$, we have

$$\frac{M_t-v^*t}{\sqrt{t}}\overset{\mathrm{d}}{\Rightarrow}\mathcal{N}(0,\widetilde{\sigma}^2_{-\lambda^*}), \qquad \text{as } t\to+\infty,$$

where $\widetilde{\sigma}^2_{-\lambda^*} := (\sigma'_{-\lambda^*})^2 v^*/(\lambda^*)^2$ and $(\sigma'_{-\lambda^*})^2$ is defined in (2.33).

(ii) If $(\sigma'_{-\lambda^*})^2>0$, then the sequence of processes

$$[0,\infty)\ni t\mapsto\frac{M_{nt}-v^*nt}{\tilde{\sigma}_{-\lambda^*}\sqrt{n}}, \quad n\in\mathbb{N}$$

converges weakly as $n\to\infty$ to a standard Brownian motion on $[0,\infty)$, in the topology of $C[0,\infty)$.

Here is a brief description of our strategy for proving the main results. Theorem 1.4 and Corollary 1.5 are proved using the spine decomposition and the long-time behavior of the spine process. To prove Theorem 1.6, we adapt some ideas from [7, Section 6] and [10, Section 4]. Roughly speaking, we define a new process $\mathcal{L}_t$ in (3.6). To prove that the sum of $\mathrm{P}_0^\xi(\inf_{n-1\le t\le n} M_t - V_n < -\Gamma \ln n)$ over n is finite, we only need to show that $\inf_{|y|\le 1} \mathrm{P}_y^\xi(\mathcal{L}_n > 0) \ge n^{-C}$ for some constant C when n is large enough (see (3.26), (3.27) and (3.28)). To use the inequality $\mathrm{P}_y^\xi(\mathcal{L}_n > 0) \ge (\mathrm{E}_y^\xi(\mathcal{L}_n))^2/\mathrm{E}_y^\xi((\mathcal{L}_n)^2)$, we need to estimate the first moment of $\mathcal{L}_n$ from below and the second moment from above. The upper estimate of the second moment is relatively easy (see Lemma 3.1), but the lower estimate of the first moment is very delicate (see Lemma 3.4). Our strategy to prove Theorem 1.7 is to first prove functional central limit theorems for V_t, see Lemma 3.7 below, and then use Theorem 1.6 to get central limit theorems for M_t. It seems that using V_t as a tool to establish invariance principle for M_t is new.

One may expect that M_t is comparable with $m^{1/2}(t)$ in the sense that $\mathbb{P}$-a.s. ω,

$$\limsup_{t\to\infty} \frac{|M_t - m^{1/2}(t)|}{\ln t} \le \Gamma, \qquad \mathrm{P}_0^\xi\text{-a.s.} \tag{1.14}$$

for some constant Γ, and then use the central limit theorems for $m^{1/2}(t)$ proved in [10, Corollary 1.5] to get Theorem 1.7. Actually this is the strategy used in [7] for branching random walk in random environment. In [7], to prove the corresponding version of (1.14) for branching random walk in random environment (see [7, Proposition 2.3]), Černý and Drewitz first defined $\overline{m}(t)$ (see [7, (2.5)]), the front of the solution to the parabolic Anderson model and also known as the breakpoint, and then proved that for $\mathbb{P}$-a.s. ω,

$$\limsup_{t\to\infty} \frac{|\overline{m}(t) - m^{1/2}(t)|}{\ln t} \le \Gamma, \qquad \mathrm{P}_0^\xi\text{-a.s.} \tag{1.15}$$

To prove equations (1.14) and (1.15), a main step is to estimate the probability $\mathrm{P}_0^\xi(M_t \ge \overline{m}(t) - C\ln t)$ (see [7, Proposition 6.9]). The most difficult part to get this estimate is [7, Lemma 6.2] where in the definition of the set $N_t^{\mathcal{L}}$ of leading particles at time t, the breakpoint inverse T_k defined in [7, (5.21)] is related to $\overline{m}(t)$. In the proof of [7, Lemma 6.2], they also need to modify T_k (see [7, Lemma 6.3]). We think that it is possible to use (1.14) and (1.15) as tools to prove our invariance principle. We prefer to use V_t, the leading partial position of the linearized problem, and compare M_t with V_t directly. In our proof, we do not need to make any modification for V_t and we think this makes things clearer.

Remark 1.8. After our paper has been submitted, Černý, Drewitz and Oswald [6, Theorem 2.1] prove that for $\mathbb{P}$-a.s. ω, $\left(M_t - m^{1/2}(t), t\ge 0; \mathrm{P}_0^\xi\right)$ is tight. Since for given ω, V_t and $m^{1/2}(t)$ are non-random under P_0^ξ, we get (1.14). Therefore, using [6, Theorem 2.1] and [10, Corollary 1.5], we can get a central limit theorem for M_t as well, but not the functional central limit theorem for M_t.

1.3 The organization of the paper

In Section 2, we first introduce the spine decomposition, and then prove some basic properties for the eigenvalues $\gamma(\lambda)$ and eigenfunctions $\phi(x,\lambda)$. At the end of this section, we prove a strong law of large number for the spine process Ξ_t and a central limit theorem for the hitting time of Ξ_t.

In Section 3, we prove the main results. In subsection 3.1, we prove Theorem 1.4 and Corollary 1.5 with the help of the spine decomposition and the long-time behavior of the spine process established in subsection 2.1 and subsection 2.3. In subsection 3.2, we prove Theorem 1.6. The method is inspired by [7, Section 6] and [10, Section 4] but we use the newly defined process V_t to make the proof more accessible. The most complicated part for the proof of Theorem 1.6 is postponed to subsection 3.4. In subsection 3.3, we prove Theorem 1.7, which is a direct corollary by Theorem 1.6 and [21, Lemma 3.1] (or see Lemma 3.6).

2 Preliminaries

2.1 Martingale change of measure and spine decomposition

It follows from the relationship between ϕ and ψ that

$$\begin{aligned} &\ln\psi(x,\lambda)=-\lambda x+\ln\phi(x,\lambda),\quad \frac{\psi_x(x,\lambda)}{\psi(x,\lambda)}=-\lambda+\frac{\phi_x(x,\lambda)}{\phi(x,\lambda)},\\ &\frac{\psi_\lambda(x,\lambda)}{\psi(x,\lambda)}=-x+\frac{\phi_\lambda(x,\lambda)}{\phi(x,\lambda)},\quad \frac{1}{x}\ln\psi(x,\lambda)\to-\lambda \text{ as } |x|\to\infty. \end{aligned} \tag{2.1}$$

For any $\omega\in\Omega_1$, using the non-negative martingale $\left(W_t(\lambda),\widetilde{\mathcal{F}}_t,t\geq 0;\mathrm{P}_x^\xi\right)$, we define a new probability measure $\mathrm{Q}_x^{\xi,\lambda}$ by

$$\left.\frac{\mathrm{d}\mathrm{Q}_x^{\xi,\lambda}}{\mathrm{d}\mathrm{P}_x^\xi}\right|_{\widetilde{\mathcal{F}}_t}(X)=\frac{W_t(\lambda)}{e^{-\lambda x}\phi(x,\lambda)}.$$

To help us understand $\{X_t,t\geq 0\}$ under $\mathrm{Q}_x^{\xi,\lambda}$, we introduce a martingale change of measure for the law Π_x of Brownian motion $\{B_t,t\geq 0\}$. For fixed $\omega\in\Omega_1$, it follows from Itô's formula that Π_x-a.s.,

$$\begin{aligned} \mathrm{d}\ln\psi(B_t,\lambda)&=\frac{\psi_x(B_t,\lambda)}{\psi(B_t,\lambda)}\mathrm{d}B_t+\frac{1}{2}\frac{\psi_{xx}(B_t,\lambda)\psi(B_t,\lambda)-(\psi_x(B_t,\lambda))^2}{\psi^2(B_t,\lambda)}\mathrm{d}t\\ &=\frac{\psi_x(B_t,\lambda)}{\psi(B_t,\lambda)}\mathrm{d}B_t+\frac{1}{2}\frac{2\left(\gamma(\lambda)-(m-1)\xi(B_t)\right)\psi^2(B_t,\lambda)-(\psi_x(B_t,\lambda))^2}{\psi^2(B_t,\lambda)}\mathrm{d}t. \end{aligned}$$

Since $\Pi_x(B_0=x)=1$,

$$\ln\frac{\psi(B_t,\lambda)}{\psi(x,\lambda)}=\int_0^t\frac{\psi_x(B_s,\lambda)}{\psi(B_s,\lambda)}\mathrm{d}B_s-\frac{1}{2}\int_0^t\left(\frac{\psi_x(B_s,\lambda)}{\psi(B_s,\lambda)}\right)^2\mathrm{d}s+\int_0^t\left(\gamma(\lambda)-(m-1)\xi(B_s)\right)\mathrm{d}s.$$

Now we define

$$\begin{aligned} \Upsilon_t:=&\exp\left\{\int_0^t\frac{\psi_x(B_s,\lambda)}{\psi(B_s,\lambda)}\mathrm{d}B_s-\frac{1}{2}\int_0^t\left(\frac{\psi_x(B_s,\lambda)}{\psi(B_s,\lambda)}\right)^2\mathrm{d}s\right\}\\ =&\exp\left\{\ln\frac{\psi(B_t,\lambda)}{\psi(x,\lambda)}-\int_0^t\left(\gamma(\lambda)-(m-1)\xi(B_s)\right)\mathrm{d}s\right\}\\ =&\exp\left\{-\gamma(\lambda)t+\int_0^t(m-1)\xi(B_s)\mathrm{d}s\right\}\frac{\psi(B_t,\lambda)}{\psi(x,\lambda)},\quad t\geq 0. \end{aligned}$$

Then $\{\Upsilon_t,t\geq 0;\Pi_x\}$ is a martingale with respect to $\sigma(B_r,r\leq t)$. In fact, by (1.7) and the Markov property of $\{B_t,t\geq 0\}$, we have that for $t,s\geq 0$,

$$\begin{aligned} &\Pi_x[\Upsilon_{t+s}|\sigma(B_r:r\leq t)]\\ &=\Upsilon_t\cdot\Pi_x\left(\exp\left\{\ln\frac{\psi(B_{t+s},\lambda)}{\psi(B_t,\lambda)}-\int_t^{t+s}\left(\gamma(\lambda)-(m-1)\xi(B_r)\right)\mathrm{d}r\right\}\middle|B_t\right)\\ &=\Upsilon_t\cdot\frac{e^{-\gamma(\lambda)s}u(s,B_t,\lambda)}{\psi(B_t,\lambda)}=\Upsilon_t. \end{aligned}$$

Define a probability measure $\widetilde{\Pi}_x^{\xi,\lambda}$ by

$$\left.\frac{\mathrm{d}\widetilde{\Pi}_x^{\xi,\lambda}}{\mathrm{d}\Pi_x}\right|_{\sigma(B_s:s\le t)} = \Upsilon_t. \quad t \ge 0. \tag{2.2}$$

Suppose that $\left\{\Xi_t, t \ge 0; \widetilde{\Pi}_x^{\xi,\lambda}\right\}$ is a diffusion such that

$$\mathrm{d}\Xi_t = \mathrm{d}W_t + \frac{\psi_x(\Xi_t,\lambda)}{\psi(\Xi_t,\lambda)}\mathrm{d}t = \mathrm{d}W_t + \left(\frac{\phi_x(\Xi_t,\lambda)}{\phi(\Xi_t,\lambda)} - \lambda\right)\mathrm{d}t,$$

where $\{W_t, t \ge 0; \widetilde{\Pi}_x^{\xi,\lambda}\}$ is a Brownian motion starting from x at time $t = 0$. Applying Girsanov's theorem, we get

$$\left\{B_t, t \ge 0; \widetilde{\Pi}_x^{\xi,\lambda}\right\} \stackrel{\mathrm{d}}{=} \left\{\Xi_t, t \ge 0; \widetilde{\Pi}_x^{\xi,\lambda}\right\}. \tag{2.3}$$

Now we define a new process $\{\widetilde{X}_t, t \ge 0\}$ with probability $\widetilde{\mathrm{P}}_x^{\xi,\lambda}$ such that $\{\widetilde{X}_t, t \ge 0; \widetilde{\mathrm{P}}_x^{\xi,\lambda}\}$ has the same law as $\{X_t, t \ge 0; \mathrm{Q}_x^{\xi,\lambda}\}$. More precisely, we consider a branching particle system in which

(i) there is an initial marked particle at $x \in \mathbb{R}$ which moves according to $(\Xi_t, \widetilde{\Pi}_x^{\xi,\lambda})$;

(ii) the branching rate of this marked particle is $m\xi(y)$ at site y;

(iii) when a marked particle dies at site y, it gives birth to k children with $\tilde{p}_k = kp_k/m, k \ge 1$;

(iv) one of these children is uniformly selected and marked, and the marked child evolves independently like its parent and the other children evolve independently with law P_y^{ξ}. Define

$$\widetilde{X}_t := \sum_{\nu\in\widetilde{N}(t)} \delta_{\widetilde{X}_t^\nu}, \quad t \ge 0,$$

where $\widetilde{N}(t)$ be the set of particles alive at time t and $\widetilde{X}_t^\nu$ is the position of particle $\nu \in \widetilde{N}(t)$. Then by [22, Theorem 2.9],

$$\{X_t, t \ge 0; \mathrm{Q}_x^{\xi,\lambda}\} \stackrel{\mathrm{d}}{=} \{\widetilde{X}_t, t \ge 0; \widetilde{\mathrm{P}}_x^{\xi,\lambda}\}.$$

The set of the marked particles along with their trajectories is called a *spine*.

2.2 Properties of $\gamma(\lambda)$ and $\phi(x,\lambda)$

We only assume that **(H1)** holds in this subsection. We will give some basic properties of $\gamma(\lambda)$ and $\phi(\cdot,\lambda)$ for $|\lambda| > \rho$. The cases that $\lambda > \rho$ and $\lambda < -\rho$ are similar. We will state our results for $|\lambda| > \rho$, but only prove the case $\lambda > \rho$. By [12, Theorem 5.1, §7.4], $\gamma(\lambda)$ is differentiable, strictly convex and $\gamma'(\lambda) > 0$ for $\lambda > \rho$, and thus $\gamma'(\lambda)$ is strictly increasing.

Lemma 2.1. *(1) $\gamma(\lambda^*) = \lambda^*\gamma'(\lambda^*)$, $v^* = \gamma'(\lambda^*)$. (2) If $|\lambda| > \lambda^*$, then $\lambda\gamma'(\lambda) > \gamma(\lambda)$ (3) if $\rho < |\lambda| < \lambda^*$, then $\lambda\gamma'(\lambda) < \gamma(\lambda)$.*

Proof. By the definition of λ^*, we know that

$$0 = \frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\frac{\gamma(\lambda)}{\lambda}\right)\bigg|_{\lambda=\lambda^*} = \frac{\gamma'(\lambda^*)\lambda^* - \gamma(\lambda^*)}{(\lambda^*)^2} \;\Leftrightarrow\; \gamma(\lambda^*) = \lambda^*\gamma'(\lambda^*).$$

For $\lambda > \rho$,

$$\begin{aligned}\lambda\gamma'(\lambda) - \gamma(\lambda) &= \lambda\gamma'(\lambda) - \gamma(\lambda) - (\lambda^*\gamma'(\lambda^*) - \gamma(\lambda^*)) \\ &= \lambda^*\left(\gamma'(\lambda) - \gamma'(\lambda^*)\right) + \int_{\lambda*}^{\lambda} \left(\gamma'(\lambda) - \gamma'(y)\right)\mathrm{d}y.\end{aligned}$$

Since $\gamma'(\lambda)$ is strictly increasing, the conclusions of the lemma follow immediately. □

Recall that $\psi(x,\lambda)=e^{-\lambda x}\phi(x,\lambda)$ satisfies (1.6). Thus, for fixed $\lambda>0$, $\psi(x,\lambda)\to 0$ as $x\to+\infty$. For any $y\in\mathbb{R}$, define

$$H_y := \inf\{t>0:\ B_t=y\}.$$

According to the Feynman-Kac formula, we have

$$\psi(x,\lambda)=\Pi_x\left(\exp\left\{\int_0^{H_y}((m-1)\xi(B_s)-\gamma(\lambda))\,\mathrm{d}s\right\}\right)\psi(y,\lambda),\quad x>y. \tag{2.4}$$

In fact for fixed $\lambda>\rho$ and any $z>y$, it is easy to see that $\psi(\cdot,\lambda)$ satisfies the condition of [9, Corollary 2] in the interval (y,z) with $q(x)=(m-1)\xi(x)-\gamma(\lambda)$. By [9, Corollary 2], the function

$$u_{(y,z)}(x):=\Pi_x\left(\exp\left\{\int_0^{H_y\wedge H_z}((m-1)\xi(B_s)-\gamma(\lambda))\,\mathrm{d}s\right\}\right)$$

is bounded in $[y,z]$. Therefore, by [9, Theorem 2.3], for any $x\in(y,z)$, it holds that

$$\begin{aligned}\psi(x,\lambda)&=\Pi_x\left(\exp\left\{\int_0^{H_y\wedge H_z}((m-1)\xi(B_s)-\gamma(\lambda))\,\mathrm{d}s\right\}\psi(B_{H_y\wedge H_z},\lambda)\right)\\&=\Pi_x\left(\exp\left\{\int_0^{H_y}((m-1)\xi(B_s)-\gamma(\lambda))\,\mathrm{d}s\right\}1_{\{H_y<H_z\}}\right)\psi(y,\lambda)\\&\quad+\Pi_x\left(\exp\left\{\int_0^{H_z}((m-1)\xi(B_s)-\gamma(\lambda))\,\mathrm{d}s\right\}1_{\{H_z<H_y\}}\right)\psi(z,\lambda).\end{aligned} \tag{2.5}$$

To prove (2.4), it suffices to show that the second term in the right-hand side of (2.5) tends to 0 as $z\to+\infty$. Note that $\psi(\cdot,-\lambda)$ also satisfies the condition of [9, Corollary 2], and thus (2.5) holds with λ replaced by $-\lambda$. Using the fact that $\gamma(\lambda)=\gamma(-\lambda)$, we get that

$$\Pi_x\left(\exp\left\{\int_0^{H_z}((m-1)\xi(B_s)-\gamma(\lambda))\,\mathrm{d}s\right\}1_{\{H_z<H_y\}}\right)\psi(z,\lambda)\le\frac{\psi(x,-\lambda)}{\psi(z,-\lambda)}\psi(z,\lambda),$$

which tends to 0 as $z\to+\infty$ by (2.1) and thus (2.4) holds.

In particular, for $x>0$,

$$\psi(x,\lambda)=\Pi_x\left(\exp\left\{\int_0^{H_0}((m-1)\xi(B_s)-\gamma(\lambda))\,\mathrm{d}s\right\}\right). \tag{2.6}$$

Recall that $\mathcal{F}_x=\sigma\{\xi(y):y\le x\}$ and $\mathcal{F}^x=\sigma\{\xi(y):y\ge x\}$. By (2.4), we have for any $x>y$,

$$\frac{\psi(x,\lambda)}{\psi(y,\lambda)}\in\mathcal{F}^y.$$

Similarly, for fixed $\lambda<0$, $\psi(x,\lambda)\to0$ as $x\to-\infty$. It follows from the Feynman-Kac formula that

$$\psi(x,\lambda)=\Pi_x\left(\exp\left\{\int_0^{H_y}((m-1)\xi(B_s)-\gamma(\lambda))\,\mathrm{d}s\right\}\right)\psi(y,\lambda),\quad x<y. \tag{2.7}$$

Thus

$$\frac{\psi(x,\lambda)}{\psi(y,\lambda)}\in\mathcal{F}_y,\quad \text{for } x<y,\lambda<0.$$

Lemma 2.2. *For all $|\lambda| > \rho$ and $\omega \in \Omega_1$, $\phi(\cdot, \lambda, \omega)$ is differentiable in x and λ.*

Proof. Without loss of generality, we assume $\lambda \in (\rho, \infty)$. We claim that, for fixed $x > 0$, ψ is differentiable as a function of λ and

$$\psi_\lambda(x, \lambda) := \frac{\partial \psi(x, \lambda)}{\partial \lambda} = -\gamma'(\lambda)\Pi_x \left(H_0 \exp\left\{ \int_0^{H_0} ((m-1)\xi(B_s) - \gamma(\lambda))\,\mathrm{d}s \right\} \right). \tag{2.8}$$

In fact, we can choose $\lambda_1 \in (\rho, \lambda)$ and $K > 0$ such that $h \exp\{-(\gamma(\lambda) - \gamma(\lambda_1))\, h\} \le K$ for all $h > 0$, and thus

$$\begin{aligned} &\Pi_x \left(H_0 \exp\left\{ \int_0^{H_0} ((m-1)\xi(B_s) - \gamma(\lambda))\,\mathrm{d}s \right\} \right) \\ &\le K\Pi_x \left(\exp\left\{ \int_0^{H_0} ((m-1)\xi(B_s) - \gamma(\lambda_1))\,\mathrm{d}s \right\} \right). \end{aligned}$$

We can now use the dominated convergence theorem in (2.6) to get (2.8).

The same argument together with (2.4) shows that, for $x > y$, $\psi(x, \lambda)/\psi(y, \lambda)$ is differentiable in $\lambda \in (\rho, \infty)$. By choosing $x > 0 \ge y$, we get that ψ is differentiable in λ for all $x \in \mathbb{R}$. The desired conclusion follows immediately from (1.5). $\square$

To compare with the constant case in which $\phi \equiv 1$, we will use ϕ in the statements of the results, but use ψ in the proof for convenience. By the uniqueness of ϕ, we have $\mathbb{P}$-almost surely $\phi(z + y, \lambda, \omega) = \phi(y, \lambda, \omega)\phi(z, \lambda, \theta_y \omega)$. Thus by (2.1), for any $z, y \in \mathbb{R}$,

$$\begin{aligned} \ln \psi(z + y, \lambda, \omega) - \ln \psi(y, \lambda, \omega) &= -\lambda z + \ln \phi(z + y, \lambda, \omega) - \ln \phi(y, \lambda, \omega) \\ &= -\lambda z + \ln \phi(z, \lambda, \theta_y \omega). \end{aligned} \tag{2.9}$$

Taking derivative with respect to λ, we get

$$\frac{\psi_\lambda(z + y, \lambda, \omega)}{\psi(z + y, \lambda, \omega)} - \frac{\psi_\lambda(y, \lambda, \omega)}{\psi(y, \lambda, \omega)} = -z + \frac{\phi_\lambda(z, \lambda, \theta_y \omega)}{\phi(z, \lambda, \theta_y \omega)}. \tag{2.10}$$

Note also that (2.9) is equivalent to

$$\frac{\psi(z + y, \lambda, \omega)}{\psi(y, \lambda, \omega)} = e^{-\lambda z}\phi(z, \lambda, \theta_y \omega). \tag{2.11}$$

Lemma 2.3. *Suppose $|\lambda| > \rho$ and $\gamma(\lambda) > (m-1)\mathrm{es}$.*

(1) For all $|\lambda| > \rho$, there exist positive constants $C_1(\lambda), C_2(\lambda)$ depending on λ only such that

$$-C_1(\lambda) \le \left(\frac{\psi_\lambda(x, \lambda)}{\psi(x, \lambda)} \right)_x = \left(-x + \frac{\phi_\lambda(x, \lambda)}{\phi(x, \lambda)} \right)_x \le -C_2(\lambda), \quad x \in \mathbb{R}. \tag{2.12}$$

(2) If $\lambda > \rho$, there exist positive constants $K_1(\lambda), K_2(\lambda)$ depending on λ only such that

$$-K_1(\lambda) \le \frac{\psi_x(x, \lambda)}{\psi(x, \lambda)} = -\lambda + \frac{\phi_x(x, \lambda)}{\phi(x, \lambda)} \le -K_2(\lambda), \quad x \in \mathbb{R}; \tag{2.13}$$

and if $\lambda < -\rho$,

$$K_2(\lambda) \le \frac{\psi_x(x, \lambda)}{\psi(x, \lambda)} = -\lambda + \frac{\phi_x(x, \lambda)}{\phi(x, \lambda)} \le K_1(\lambda), \quad x \in \mathbb{R}.$$

(3)

$$\lim_{|x| \to \infty} \frac{\phi_\lambda(x, \lambda)}{x\phi(x, \lambda)} = 0, \qquad \mathbb{P}\text{-a.s.}.$$

(4) $\gamma'(\lambda)$ is continuous in the open set $\{\lambda \in \mathbb{R} : |\lambda| > \rho, \gamma(\lambda) > (m-1)\mathrm{es}\}$.

Proof. Recall that, by (2.1), $\frac{\psi_\lambda(x,\lambda)}{\psi(x,\lambda)} = -x + \frac{\phi_\lambda(x,\lambda)}{\phi(x,\lambda)}$. Put $g(x,\lambda) := \frac{\psi_\lambda(x,\lambda)}{\psi(x,\lambda)}$ for simplicity.

(1) Fix $\lambda > \rho$. For $x > y$, taking logarithm and then differentiating in λ in (2.4), we get

$$\frac{\psi_\lambda(x,\lambda)}{\psi(x,\lambda)} = -\gamma'(\lambda)\frac{\Pi_x\left(\exp\left\{\int_0^{H_y}((m-1)\xi(B_s)-\gamma(\lambda))\,\mathrm{d}s\right\}H_y\right)}{\Pi_x\left(\exp\left\{\int_0^{H_y}((m-1)\xi(B_s)-\gamma(\lambda))\,\mathrm{d}s\right\}\right)} + \frac{\psi_\lambda(y,\lambda)}{\psi(y,\lambda)}. \tag{2.14}$$

Hence, when $x > y$,

$$\begin{aligned} g(x,\lambda) - g(y,\lambda) &= -\gamma'(\lambda)\frac{\Pi_x\left(\exp\left\{\int_0^{H_y}((m-1)\xi(B_s)-\gamma(\lambda))\,\mathrm{d}s\right\}H_y\right)}{\Pi_x\left(\exp\left\{\int_0^{H_y}((m-1)\xi(B_s)-\gamma(\lambda))\,\mathrm{d}s\right\}\right)} \\ &\le -\gamma'(\lambda)\frac{\Pi_x\left(\exp\left\{((m-1)\mathrm{ei}-\gamma(\lambda))H_y\right\}H_y\right)}{\Pi_x\left(\exp\left\{((m-1)\mathrm{es}-\gamma(\lambda))H_y\right\}\right)} \\ &= \frac{-\gamma'(\lambda)(x-y)}{\sqrt{2(\gamma(\lambda)-(m-1)\mathrm{ei})}}\cdot\frac{e^{-(x-y)\sqrt{2(\gamma(\lambda)-(m-1)\mathrm{ei})}}}{e^{-(x-y)\sqrt{2(\gamma(\lambda)-(m-1)\mathrm{es})}}}. \end{aligned} \tag{2.15}$$

In the first inequality above we used the fact that $-\gamma'(\lambda) < 0$ and in last equality the fact that for any $u > 0$ and $x, y \in \mathbb{R}$,

$$\Pi_x\left(e^{-uH_y}\right) = e^{-|y-x|\sqrt{2u}} \qquad \text{and} \qquad \Pi_x\left(H_y e^{-uH_y}\right) = \frac{|y-x|}{\sqrt{2u}}e^{-|y-x|\sqrt{2u}}. \tag{2.16}$$

Dividing both sides of (2.15) by $x-y$ and letting $y \uparrow x$, we get

$$g_x(x,\lambda) \le \frac{-\gamma'(\lambda)}{\sqrt{2(\gamma(\lambda)-(m-1)\mathrm{ei})}} =: -C_2(\lambda), \quad x \in \mathbb{R}.$$

Similarly, for $x > y$,

$$\begin{aligned} g(x,\lambda) - g(y,\lambda) &\ge -\gamma'(\lambda)\frac{\Pi_x\left(\exp\left\{((m-1)\mathrm{es}-\gamma(\lambda))H_y\right\}H_y\right)}{\Pi_x\left(\exp\left\{((m-1)\mathrm{ei}-\gamma(\lambda))H_y\right\}\right)} \\ &= \frac{-\gamma'(\lambda)(x-y)}{\sqrt{2(\gamma(\lambda)-(m-1)\mathrm{es})}}\cdot\frac{e^{-(x-y)\sqrt{2(\gamma(\lambda)-(m-1)\mathrm{es})}}}{e^{-(x-y)\sqrt{2(\gamma(\lambda)-(m-1)\mathrm{ei})}}}, \end{aligned}$$

and thus

$$g_x(x,\lambda) \ge \frac{-\gamma'(\lambda)}{\sqrt{2(\gamma(\lambda)-(m-1)\mathrm{es})}} =: -C_1(\lambda), \quad x \in \mathbb{R}.$$

Hence (2.12) is valid.

When $\lambda < -\rho$, (2.14) holds for all $x < y$. In this case we have that for $x < y$,

$$g(y,\lambda) - g(x,\lambda) = \gamma'(\lambda)\frac{\Pi_x\left(\exp\left\{\int_0^{H_y}((m-1)\xi(B_s)-\gamma(\lambda))\,\mathrm{d}s\right\}H_y\right)}{\Pi_x\left(\exp\left\{\int_0^{H_y}((m-1)\xi(B_s)-\gamma(\lambda))\,\mathrm{d}s\right\}\right)} < 0.$$

An argument similar as above shows that (2.12) holds.

(2) The argument is very similar to that of (1). We only prove the case of $\lambda > \rho$. By (2.4), for $x > y$ we have

$$\int_y^x \frac{\psi_x(z,\lambda)}{\psi(z,\lambda)}\mathrm{d}z = \ln\psi(x,\lambda) - \ln\psi(y,\lambda) = \ln\Pi_x\left(\exp\left\{\int_0^{H_y}((m-1)\xi(B_s)-\gamma(\lambda))\,\mathrm{d}s\right\}\right)$$

$$\leq \ln \Pi_x \left(\exp\left\{ \int_0^{H_y} ((m-1)\mathrm{es} - \gamma(\lambda))\, \mathrm{d}s \right\} \right) = -(x-y)\sqrt{2\left(\gamma(\lambda) - (m-1)\mathrm{es}\right)}.$$

Similarly, for any $x > y$,

$$\int_y^x \frac{\psi_x(z,\lambda)}{\psi(z,\lambda)} \mathrm{d}z \geq -(x-y)\sqrt{2\left(\gamma(\lambda) - (m-1)\mathrm{ei}\right)}.$$

Letting $K_2(\lambda) := \sqrt{2\left(\gamma(\lambda) - (m-1)\mathrm{es}\right)}$ and $K_1(\lambda) := \sqrt{2\left(\gamma(\lambda) - (m-1)\mathrm{ei}\right)}$, we get the results of (2).

(3) For $\lambda > \rho$, in view of (1), for any $x, y \in \mathbb{R}$ with $|x-y| \leq 1$,

$$|g(x,\lambda) - g(y,\lambda)| = \left| \frac{\psi_\lambda(x,\lambda)}{\psi(x,\lambda)} - \frac{\psi_\lambda(y,\lambda)}{\psi(y,\lambda)} \right| = \left| \frac{\phi_\lambda(x,\lambda)}{\phi(x,\lambda)} - \frac{\phi_\lambda(y,\lambda)}{\phi(y,\lambda)} \right| \leq C_1(\lambda). \tag{2.17}$$

Thus, it suffices to prove that for $k \in \mathbb{Z}$,

$$\lim_{|k| \to \infty} \frac{g(k,\lambda)}{k} = -1, \qquad \mathbb{P}\text{-a.s.}.$$

By (2.10) and (2.17), $\{g(k+1,\lambda,\omega) - g(k,\lambda,\omega) : k \in \mathbb{Z}\}$ is a stationary and ergodic sequence with uniform bound $C_1(\lambda)$. By (2.9) and (2.13), we also have that $\{\ln \psi(k+1,\lambda,\omega) - \ln \psi(k,\lambda,\omega) : k \in \mathbb{Z}\}$ is a stationary and ergodic sequence with uniform bound $K_1(\lambda)$. Thus,

$$\mathbb{E} \ln \psi(1,\lambda,\omega) = \lim_{k \to \infty} \frac{\ln \psi(k,\lambda,\omega)}{k} = -\lambda.$$

Taking derivative with respect to λ in the display above, using the boundness of g, we get that

$$\mathbb{E} g(1,\lambda,\omega) = -1. \tag{2.18}$$

By Birkhoff's ergodic theorem,

$$\lim_{|k| \to \infty} \frac{g(k,\lambda)}{k} = \mathbb{E} g(1,\lambda,\omega) = -1, \qquad \mathbb{P}\text{-a.s.}.$$

This completes the proof of (3).

(4) Suppose $\lambda > \rho$ and $\gamma(\lambda) > (m-1)\mathrm{es}$. Using (2.14), we have

$$g(1,\lambda) = \frac{\psi_\lambda(1,\lambda)}{\psi(1,\lambda)} = -\gamma'(\lambda) \left(\frac{\Pi_1 \left(\exp\left\{ \int_0^{H_0} ((m-1)\xi(B_s) - \gamma(\lambda))\, \mathrm{d}s \right\} H_0 \right)}{\Pi_1 \left(\exp\left\{ \int_0^{H_0} ((m-1)\xi(B_s) - \gamma(\lambda))\, \mathrm{d}s \right\} \right)} \right).$$

Taking expectation with respect to $\mathbb{P}$, and using (2.18), we get

$$1 = \gamma'(\lambda) \mathbb{E} \left(\frac{\Pi_1 \left(\exp\left\{ \int_0^{H_0} ((m-1)\xi(B_s) - \gamma(\lambda))\, \mathrm{d}s \right\} H_0 \right)}{\Pi_1 \left(\exp\left\{ \int_0^{H_0} ((m-1)\xi(B_s) - \gamma(\lambda))\, \mathrm{d}s \right\} \right)} \right).$$

Using the bounded convergence theorem and the continuity of $\gamma(\lambda)$, we get $\gamma'(\lambda)$ is continuous in λ. □

Remark 2.4. By Lemma 2.3(2), if **(H2)** holds, then $\psi(\cdot, -\lambda^*)$ is increasing and for any $y < z$,

$$\ln \psi(z, -\lambda^*) - \ln \psi(y, -\lambda^*) = \int_y^z (\ln \psi(x, -\lambda^*))_x \, \mathrm{d}x \in \left[K_2(-\lambda^*)(z-y), K_1(-\lambda^*)(z-y) \right].$$

Thus, for any $y < z$,

$$K_2(-\lambda^*)(z-y) \le \ln\psi(z,-\lambda^*) - \ln\psi(y,-\lambda^*) \le K_1(-\lambda^*)(z-y). \tag{2.19}$$

Recalling the definition of V_t in (1.13), and using (2.19), we get that for any $s < t$,

$$\frac{\gamma(\lambda^*)(t-s)}{K_1(-\lambda^*)} \le V_t - V_s \le \frac{\gamma(\lambda^*)(t-s)}{K_2(-\lambda^*)}. \tag{2.20}$$

Throughout this paper, for any real x, we use $[x]$ to denote the integer part of x, and $\lceil x \rceil$ to denote the smallest integer larger than x. The following lemma is from [13, Theorem 5.5], see also [21, Theorem 2.8].

Lemma 2.5. *Suppose that $\{\eta_k\}_{k=0}^{\infty} \subset L^2(\Omega,\mathcal{F},\mathbb{P})$ is a stationary sequence with $\mathbb{E}(\eta_k)=0$ and that*

$$\sum_{k=1}^{\infty}\left(\eta_0 - \mathbb{E}(\eta_0|\mathcal{F}_k)\right), \quad \textit{and} \quad \sum_{k=1}^{\infty}\mathbb{E}(\eta_k|\mathcal{F}_0)$$

converge in $L^2(\Omega,\mathcal{F},\mathbb{P})$. Then the limit

$$\sigma^2 := \lim_{N\to\infty}\frac{1}{N}\mathbb{E}\left(\sum_{i=0}^{N-1}\eta_k\right)^2$$

exists and is finite. If $\sigma^2>0$, then

$$\frac{\sum_{i=0}^{N-1}\eta_k}{\sqrt{N}} \overset{\mathrm{d}}{\Longrightarrow} \mathcal{N}(0,\sigma^2)$$

and the process

$$\frac{1}{\sigma\sqrt{n}}\left(\sum_{i=0}^{[nt]-1}\eta_k + (nt-[nt])\eta_{[nt]}\right), \qquad t\in[0,1],$$

converges weakly to a standard Brownian motion on $[0,1]$.

Lemma 2.6. *Suppose $|\lambda|>\rho$ and $\gamma(\lambda)>(m-1)\mathrm{es}$, then there exists $\sigma_\lambda^2\ge 0$ such that under $\mathbb{P}$,*

$$\frac{\ln\phi(x,\lambda)-\lambda\frac{\phi_\lambda(x,\lambda)}{\phi(x,\lambda)}}{\sqrt{|x|}} \overset{\mathrm{d}}{\Longrightarrow} \mathcal{N}(0,\sigma_\lambda^2), \qquad \text{as } |x|\to\infty. \tag{2.21}$$

Moreover, if $\sigma_\lambda^2>0$, then the process

$$\frac{\ln\phi(nt,\lambda)-\lambda\frac{\phi_\lambda(nt,\lambda)}{\phi(nt,\lambda)}}{\sigma_\lambda\sqrt{n}}, \quad t\in[0,\infty),$$

converge weakly to a standard Brownian motion on $[0,\infty)$ in the Skorohod topology and

$$\limsup_{|x|\to+\infty}\frac{\ln\phi(x,\lambda)-\lambda\frac{\phi_\lambda(x,\lambda)}{\phi(x,\lambda)}}{\sqrt{|x|}} = +\infty, \qquad \mathbb{P}\text{-a.s.}.$$

If $\sigma_\lambda^2=0$, then

$$\sup_{x\in\mathbb{R}}\left\|\ln\phi(x,\lambda)-\lambda\frac{\phi_\lambda(x,\lambda)}{\phi(x,\lambda)}\right\|_\infty < \infty.$$

Proof. By (2.1), we have that

$$\ln\phi(x,\lambda)-\lambda\frac{\phi_\lambda(x,\lambda)}{\phi(x,\lambda)}=\ln\psi(x,\lambda)-\lambda\frac{\psi_\lambda(x,\lambda)}{\psi(x,\lambda)} \tag{2.22}$$

holds for all $|\lambda|>\rho$ and $x\in\mathbb{R}$. In view of Lemma 2.3(1)-(2), to prove (2.21), it suffices to prove it for the case of $x\in\mathbb{Z}$.

Fix $\lambda>\rho$. Let $x=n\in\mathbb{Z}$ and define

$$U_i:=\ln\psi(i+1,\lambda,\omega)-\ln\psi(i,\lambda,\omega)-\lambda\left(\frac{\psi_\lambda(i+1,\lambda,\omega)}{\psi(i+1,\lambda,\omega)}-\frac{\psi_\lambda(i,\lambda,\omega)}{\psi(i,\lambda,\omega)}\right),\quad i\in\mathbb{Z}. \tag{2.23}$$

Then by (2.12) and (2.13), $\{U_i\}_{i\in\mathbb{Z}}$ is a stationary and ergodic sequence with uniform bound $C_1(\lambda)+\lambda K_1(\lambda)$, and

$$\ln\psi(n,\lambda)-\lambda\frac{\psi_\lambda(n,\lambda)}{\psi(n,\lambda)}=\sum_{i=0}^{n-1}U_i,\qquad \ln\psi(-n,\lambda)-\lambda\frac{\psi_\lambda(-n,\lambda)}{\psi(-n,\lambda)}=\sum_{i=1}^{n}-U_{-i}.$$

Since

$$\ln\psi(n,\lambda)-\lambda\frac{\psi_\lambda(n,\lambda)}{\psi(n,\lambda)}\overset{\mathrm{d}}{=}\lambda\frac{\psi_\lambda(-n,\lambda)}{\psi(-n,\lambda)}-\ln\psi(-n,\lambda)$$

by the stationarity of U_i, we only need to show (2.21) for $x=n\in\mathbb{Z}^+$. By Lemma 2.3(3) and Birkhoff's ergodic theorem, $\mathbb{P}$-almost surely,

$$\mathbb{E}U_0=\lim_{n\to+\infty}\frac{\sum_{i=0}^{n-1}U_i}{n}=\lim_{n\to+\infty}\left(\frac{\ln\psi(n,\lambda)}{n}-\lambda\frac{\psi_\lambda(n,\lambda)}{n\psi(n,\lambda)}\right)=-\lambda+\lambda=0.$$

We will prove (2.21) using Lemma 2.5. First we check the conditions of Lemma 2.5 are satisfied using the argument of [10, Lemma A.2]. For $j\ge 2$, set

$$A_1(\lambda)=A_1(\lambda,\omega;j):=\Pi_1\left(\exp\left\{\int_0^{H_0}((m-1)\xi(B_s)-\gamma(\lambda))\,\mathrm{d}s\right\};\sup_{0\le s\le H_0}(B_s-1)<[j/2]\right);$$
$$A_2(\lambda)=A_2(\lambda,\omega;j):=\Pi_1\left(\exp\left\{\int_0^{H_0}((m-1)\xi(B_s)-\gamma(\lambda))\,\mathrm{d}s\right\};\sup_{0\le s\le H_0}(B_s-1)\ge[j/2]\right)$$

and $A_i'(\lambda):=\frac{\partial A_i(\lambda)}{\partial\lambda}$ for $i=1,2$. According to (2.4) and (2.14), we have

$$U_0=\ln\left(A_1(\lambda)+A_2(\lambda)\right)-\lambda\frac{A_1'(\lambda)}{A_1(\lambda)+A_2(\lambda)}-\lambda\frac{A_2'(\lambda)}{A_1(\lambda)+A_2(\lambda)}.$$

Note that $A_1(\lambda)$ is $\mathcal{F}_{[j/2]+1}\cap\mathcal{F}^0$- measurable. By the assumption that $\gamma(\lambda)>(m-1)\mathrm{es}$, we have

$$A_1(\lambda)\le\Pi_1\left(\exp\left\{\left((m-1)\mathrm{es}-\gamma(\lambda)\right)H_0\right\}\right)=:c_1(\lambda)<1,$$

and for all $j\ge 2$,

$$\begin{aligned}A_1(\lambda)&\ge\Pi_1\left(\exp\left\{\left((m-1)\mathrm{ei}-\gamma(\lambda)\right)H_0\right\};\sup_{0\le s\le H_0}(B_s-1)<[j/2]\right)\\&\ge\Pi_1\left(\exp\left\{\left((m-1)\mathrm{ei}-\gamma(\lambda)\right)H_0\right\};\sup_{0\le s\le H_0}(B_s-1)<1\right)=:c_2(\lambda)>0.\end{aligned}$$

Therefore, for $j\ge 2$, we have

$$0<c_2(\lambda)\le A_1(\lambda)\le c_1(\lambda)<1. \tag{2.24}$$

Next, if $H_0 > j$, then

$$\exp\left\{\int_0^{H_0}((m-1)\xi(B_s)-\gamma(\lambda))\,\mathrm{d}s\right\} \le \exp\left\{((m-1)\mathrm{es}-\gamma(\lambda))\,j\right\};$$

if $H_0 \le j$, then $\exp\left\{\int_0^{H_0}((m-1)\xi(B_s)-\gamma(\lambda))\,\mathrm{d}s\right\} \le 1$, and

$$\left\{\sup_{0\le s\le H_0}(B_s-1)\ge [j/2]\right\} \subset \left\{\sup_{0\le s\le j}(B_s-1)\ge [j/2]\right\}.$$

Thus,

$$0 \le A_2(\lambda) \le \exp\left\{((m-1)\mathrm{es}-\gamma(\lambda))\,j\right\} + \Pi_1\left(\sup_{0\le s\le j}(B_s-1)\ge [j/2]\right).$$

By the reflection principle and Markov's inequality, there exists a positive constant $\delta < (\gamma(\lambda)-(m-1)\mathrm{es})\wedge \frac{1}{8}$ such that for all $j \ge 2$,

$$\begin{aligned} 0 \le A_2(\lambda) &\le e^{-j\delta} + 2\Pi_0(B_j \ge [j/2]) \le e^{-j\delta} + 2\frac{\Pi_0\left(e^{B_j/2}\right)}{e^{[j/2]/2}} \\ &\le e^{-j\delta} + 2e^{j/8-(j/2-1)/2} \le 5e^{-j\delta}. \end{aligned} \tag{2.25}$$

Since $\mathcal{F}_{[j/2]+1} \subset \mathcal{F}_j$ for $j \ge 2$, we have $A_1(\lambda) \in \mathcal{F}_j$. Thus

$$\begin{aligned} &\left|\ln\phi(1,\lambda) - \mathbb{E}\left(\ln\phi(1,\lambda)\middle|\mathcal{F}_j\right)\right| = \left|\ln\psi(1,\lambda) - \mathbb{E}\left(\ln\psi(1,\lambda)\middle|\mathcal{F}_j\right)\right| \\ &= \left|\ln(A_1(\lambda)+A_2(\lambda)) - \mathbb{E}\left(\ln(A_1(\lambda)+A_2(\lambda))\middle|\mathcal{F}_j\right)\right| \\ &\le \left|\ln(A_1(\lambda)) - \mathbb{E}\left(\ln(A_1(\lambda))\middle|\mathcal{F}_j\right)\right| + \frac{10}{c_2(\lambda)}e^{-j\delta} = \frac{10}{c_2(\lambda)}e^{-j\delta}, \end{aligned} \tag{2.26}$$

where in the last inequality we used the following estimate:

$$0 \le \ln\left(1+\frac{A_2(\lambda)}{A_1(\lambda)}\right) \le \frac{A_2(\lambda)}{A_1(\lambda)} \le \frac{5}{c_2(\lambda)}e^{-j\delta}.$$

Take $\alpha > 0$ small so that $\gamma(\lambda) > (m-1)\mathrm{es}+\alpha$. Noticing that $\sup_{h>0}(h\exp\{-\alpha h\}) < \infty$, we have that there is a constant c_3 depending only on α such that $h\exp\{-\alpha h\} \le c_3, \forall h > 0$ and thus

$$\begin{aligned} 0 \le -A_1'(\lambda) &\le c_3\gamma'(\lambda)\Pi_1\left(e^{\int_0^{H_0}((m-1)\xi(B_s)-\gamma(\lambda)+\alpha)\mathrm{d}s}; \sup_{0\le s\le H_0}(B_s-1) < [j/2]\right), \\ 0 \le -A_2'(\lambda) &\le c_3\gamma'(\lambda)\Pi_1\left(e^{\int_0^{H_0}((m-1)\xi(B_s)-\gamma(\lambda)+\alpha)\mathrm{d}s}; \sup_{0\le s\le H_0}(B_s-1) \ge [j/2]\right). \end{aligned}$$

Define $\gamma(\tilde{\lambda}) := \gamma(\lambda)-\alpha$. Then we get

$$0 \le -A_i'(\lambda) \le c_3\gamma'(\lambda)A_i(\tilde{\lambda}), \quad i = 1,2.$$

By (2.25), there exists $\delta \in \left(0, (\gamma(\tilde{\lambda})-(m-1)\mathrm{es}-\alpha)\wedge\frac{1}{8}\right)$ such that $0 \le A_2(\tilde{\lambda}) \le 5e^{-j\delta}$, and thus

$$0 \le -A_1'(\lambda) \le c_3\gamma'(\lambda)c_1(\tilde{\lambda}), \quad 0 \le -A_2'(\lambda) \le 5c_3\gamma'(\lambda)e^{-j\delta}. \tag{2.27}$$

In the display below, $A_j(\lambda)$ and $A_j'(\lambda)$ will be simply denoted as A_j and A_j' for $j = 1,2$. Hence,

$$\left|\frac{\phi_\lambda(1,\lambda)}{\phi(1,\lambda)} - \mathbb{E}\left(\frac{\phi_\lambda(1,\lambda)}{\phi(1,\lambda)}\middle|\mathcal{F}_j\right)\right| = \left|\frac{\psi_\lambda(1,\lambda)}{\psi(1,\lambda)} - \mathbb{E}\left(\frac{\psi_\lambda(1,\lambda)}{\psi(1,\lambda)}\middle|\mathcal{F}_j\right)\right|$$

$$
\begin{aligned}
&= \left|-\lambda\frac{A_1'}{A_1+A_2} - \lambda\frac{A_2'}{A_1+A_2} - \mathbb{E}\left(-\lambda\frac{A_1'}{A_1+A_2} - \lambda\frac{A_2'}{A_1+A_2}\Big|\mathcal{F}_j\right)\right| \\
&\le \lambda\left|\frac{A_1'}{A_1+A_2} - \mathbb{E}\left(\frac{A_1'}{A_1+A_2}\Big|\mathcal{F}_j\right)\right| + \lambda\left|\frac{A_2'}{A_1+A_2}\right| + \lambda\mathbb{E}\left(\left|\frac{A_2'}{A_1+A_2}\right|\Big|\mathcal{F}_j\right) \\
&\le \lambda\frac{|A_1'|}{A_1}\left\{1-\frac{A_1}{A_1+A_2} + \mathbb{E}\left(1-\frac{A_1}{A_1+A_2}\Big|\mathcal{F}_j\right)\right\} + 10\lambda c_3\gamma'(\lambda)\frac{e^{-j\delta}}{c_2(\lambda)} \\
&\le 10\lambda c_3\gamma'(\lambda)\frac{c_1(\tilde{\lambda})}{c_2(\lambda)}\frac{e^{-j\delta}}{c_2(\lambda)} + 10\lambda c_3\gamma'(\lambda)\frac{e^{-j\delta}}{c_2(\lambda)} =: c_4e^{-j\delta},
\end{aligned} \tag{2.28}
$$

here in the second inequality above we used the fact that A_1 and A_1' are $\mathcal{F}_j$-measurable, and in the last inequality we used (2.24), (2.27), and the fact that $A_2 \ge 0$.

Now (2.26) and (2.28) imply that there exists a non-random constant c_5 independent of j such that for all $j \ge 2$,

$$
\left|\mathbb{E}\left(U_0\middle|\mathcal{F}_j\right) - U_0\right| \le c_5e^{-j\delta}, \tag{2.29}
$$

which implies that $\sum_{j=1}^{\infty}\left(\mathbb{E}\left(U_0\middle|\mathcal{F}_j\right) - U_0\right)$ converges in $L^2(\Omega,\mathcal{F},\mathbb{P})$.

Next, we estimate $\left|\mathbb{E}\left(U_j\middle|\mathcal{F}_0\right)\right|$ for $j\ge 1$. Note that when $\lambda > \rho$, $U_j \in \mathcal{F}^j$ by (2.4). Using the ζ-mixing condition and the fact that $\mathbb{E}U_j = 0$, we have

$$
|\mathbb{E}(U_j|\mathcal{F}_0)| \le \mathbb{E}\left(|U_j|\right)\zeta(j) \le \|U_0\|_\infty\zeta(j). \tag{2.30}
$$

Here and later in this proof, we use $\|\cdot\|_\infty$ to denote the supremum over ω. By the triangle inequality,

$$
\sqrt{\mathbb{E}\left(\sum_{j=m}^{n}\mathbb{E}(U_j|\mathcal{F}_0)\right)^2} \le \sum_{k=m}^{n}\sqrt{\mathbb{E}\left((\mathbb{E}(U_j|\mathcal{F}_0))^2\right)} \le \sum_{j=m}^{n}\|U_0\|_\infty\zeta(j),
$$

which implies that $\sum_{j=1}^{\infty}\mathbb{E}(U_j|\mathcal{F}_0)$ converges in $L^2(\Omega,\mathcal{F},\mathbb{P})$. Applying Lemma 2.5 and noting that $|\mathbb{E}(U_0U_j)| \le \mathbb{E}\left(|U_0|\,|\mathbb{E}\left(U_j|\mathcal{F}_0\right)|\right) \le \|U_0\|_\infty^2\zeta(j)$, we get

$$
\sigma_\lambda^2 := \lim_{N\to\infty}\frac{1}{N}\mathbb{E}\left(\sum_{i=0}^{N-1}U_i\right)^2 = \mathbb{E}(U_0^2) + 2\sum_{j=1}^{\infty}\mathbb{E}(U_0U_j) \le \|U_0\|_\infty^2 + 2\|U_0\|_\infty^2\sum_{j=1}^{\infty}\zeta(j) < \infty.
$$

Thus

$$
\frac{\ln\phi(n,\lambda) - \lambda\frac{\phi_\lambda(n,\lambda)}{\phi(n,\lambda)}}{\sqrt{n}} = \frac{\ln\psi(n,\lambda) - \lambda\frac{\psi_\lambda(n,\lambda)}{\psi(n,\lambda)}}{\sqrt{n}} \overset{\mathrm{d}}{\Longrightarrow} \mathcal{N}(0,\sigma_\lambda^2), \qquad \text{as } n\to+\infty,
$$

which concludes the proof of the first part.

If $\sigma_\lambda^2 > 0$, set $R_x = \ln\psi(x,\lambda) - \lambda\frac{\psi_\lambda(x,\lambda)}{\psi(x,\lambda)}$. For any $K>0$, by the central limit theorem for R_n we have

$$
\mathbb{P}\left(\limsup_{x\to+\infty}\frac{R_x}{\sqrt{x}} > K\right) = \lim_{x\to+\infty}\mathbb{P}\left(\sup_{y\ge x}\frac{R_y}{\sqrt{y}} > K\right) \ge \lim_{x\to\infty}\mathbb{P}\left(\frac{R_x}{\sqrt{x}} > K\right) > 0.
$$

Since $\{\limsup_{x\to+\infty}R_x/\sqrt{x} > K\}$ is an invariant set, we have

$$
\mathbb{P}\left(\limsup_{x\to\infty}\frac{R_x}{\sqrt{x}} = +\infty\right) = 1.
$$

When $x\to-\infty$, a similar argument also shows that $\mathbb{P}(\limsup_{x\to-\infty}R_x/\sqrt{|x|} = +\infty) = 1$.

For any $M > 0$, we modify the definition of U_i in (2.23) by

$$U_i = \ln \psi\left(M(i+1), \lambda, \omega\right) - \ln \psi\left(Mi, \lambda, \omega\right) - \lambda \left(\frac{\psi_\lambda\left(M(i+1), \lambda, \omega\right)}{\psi\left(M(i+1), \lambda, \omega\right)} - \frac{\psi_\lambda\left(Mi, \lambda, \omega\right)}{\psi\left(Mi, \lambda, \omega\right)} \right).$$

By Lemma 2.5, the process

$$\eta_t^n := \frac{\ln \phi(nt, \lambda) - \lambda \frac{\phi_\lambda(nt,\lambda)}{\phi(nt,\lambda)}}{\sigma_\lambda \sqrt{n}}, \quad t \in [0, M],$$

converges weakly to a standard Brownian motion on $[0, M]$. Since M is arbitrary, by [2, Lemma 3, p.173], we get the weak convergence of $\{\eta_t^n, t \geq 0\}$ to a Brownian motion on $[0, \infty)$.

If $\sigma_\lambda^2 = 0$, then by [13, §5.4] or [2, (19.8), p.198], we may write the increments $\{U_i, i \in \mathbb{Z}\}$ of the stationary ergodic process $\{\ln \psi(n, \lambda) - \lambda \frac{\psi_\lambda(n,\lambda)}{\psi(n,\lambda)}, n \in \mathbb{Z}\}$ as:

$$U_i = Y_i + (T^*)^i Z_0 - (T^*)^{i+1} Z_0, \quad i \in \mathbb{Z},$$

here $\{Y_i, i \in \mathbb{Z}\}$ is a stationary ergodic sequence of martingale differences with $Y_i \in \mathcal{F}_i$ and $\mathbb{E}\left(Y_{i+1} | \mathcal{F}_i\right) = 0$, Z_0 is a random variable and T^* is the unitary operator associated with θ_1, that is, $T^* Z_0(\omega) = Z_0(\theta_1 \omega)$. In fact, we may take

$$Z_0 = \sum_{k=0}^{\infty} \mathbb{E}\left(U_k \middle| \mathcal{F}_{-1}\right) - \sum_{k=1}^{\infty} \left(U_{-k} - \mathbb{E}\left(U_{-k} \middle| \mathcal{F}_{-1}\right)\right), \tag{2.31}$$

$Y_i = (T^*)^i Y_0$ with

$$Y_0 = \sum_{l=-\infty}^{\infty} \left(\mathbb{E}(U_l | \mathcal{F}_0) - \mathbb{E}(U_l | \mathcal{F}_{-1})\right),$$

see [13, (5.17) and (5.18), p.137]. From the proof of [13, Theorem 5.5], we know that $\mathbb{E}[Y_0^2] = \sigma_\lambda^2 = 0$ (see [13, the fourth paragraph, on p.142]), and thus $Y_i = 0$, $\mathbb{P}$-a.s. for any $i \in \mathbb{Z}$. Note that

$$\|U_{-k} - \mathbb{E}(U_{-k} | \mathcal{F}_{-1})\|_\infty = \|(T^*)^{-k}\left(U_0 - \mathbb{E}(U_0 | \mathcal{F}_{k-1})\right)\|_\infty = \|U_0 - \mathbb{E}(U_0 | \mathcal{F}_{k-1})\|_\infty.$$

Thus by (2.29),

$$\|U_{-k} - \mathbb{E}(U_{-k} | \mathcal{F}_{-1})\|_\infty \leq c_5 e^{-(k-1)\delta}, \quad k \geq 3. \tag{2.32}$$

Now using (2.32), (2.30) and (2.31), we have that

$$\|Z_0\|_\infty \leq \|U_0\|_\infty \sum_{k=0}^{\infty} \zeta(k+1) + c_5 \sum_{k=2}^{\infty} e^{-k\delta} + 4\|U_0\|_\infty < \infty.$$

Therefore, for all $n \in \mathbb{Z}^+$,

$$\left| \ln \phi(n, \lambda) - \lambda \frac{\phi_\lambda(n, \lambda)}{\phi(n, \lambda)} \right| = \left| \sum_{i=0}^{n-1} U_i \right| = |Z_0 - (T^*)^n Z_0| \leq 2\|Z_0\|_\infty,$$

which implies that

$$\sup_{x \in \mathbb{R}} \left\| \ln \phi(x, \lambda) - \lambda \frac{\phi_\lambda(x, \lambda)}{\phi(x, \lambda)} \right\|_\infty \leq 2\|Z_0\|_\infty + C_1(\lambda) + \lambda K_1(\lambda) < \infty. \qquad \square$$

Remark 2.7. Suppose that $|\lambda|>\rho$ and $\gamma(\lambda)>(m-1)\mathrm{es}$. By the proof of Lemma 2.6, and using (2.26) and (2.28), the limits

$$(\sigma'_\lambda)^2 := \lim_{x\to+\infty}\frac{1}{|x|}\mathbb{E}\left(\ln\psi(x,\lambda)+\lambda x\right)^2, \qquad (\sigma''_\lambda)^2 := \lim_{x\to+\infty}\frac{1}{|x|}\mathbb{E}\left(\frac{\psi_\lambda(x,\lambda)}{\psi(x,\lambda)}+x\right)^2, \tag{2.33}$$

exist under $\mathbb{P}$, and as $|x|\to\infty$,

$$\frac{\ln\phi(x,\lambda)}{\sqrt{|x|}} = \frac{\ln\psi(x,\lambda)+\lambda x}{\sqrt{|x|}} \overset{\mathrm{d}}{\Longrightarrow} \mathcal{N}(0,(\sigma'_\lambda)^2), \quad \frac{\phi_\lambda(x,\lambda)}{\phi(x,\lambda)\sqrt{|x|}} = \frac{\frac{\psi_\lambda(x,\lambda)}{\psi(x,\lambda)}+x}{\sqrt{|x|}} \overset{\mathrm{d}}{\Longrightarrow} \mathcal{N}(0,(\sigma''_\lambda)^2).$$

If $(\sigma'_\lambda)^2>0$, the process

$$\frac{\ln\phi(nt,\lambda)}{\sigma'_\lambda\sqrt{n}}, \quad t\in[0,\infty),$$

converge weakly to a standard Brownian motion on $[0,\infty)$ in the Skorohod topology. If $(\sigma'_\lambda)^2=0$, then $\sup_{x\in\mathbb{R}}\|\phi(x,\lambda)\|_\infty<\infty$. The same result also holds for $\frac{\phi_\lambda(x,\lambda)}{\phi(x,\lambda)}$.

2.3 Limit theorems for the spine

Define

$$T_k := \inf\{t>0 : \Xi_t-\Xi_0=k\}, \qquad k\in\mathbb{Z}.$$

Proposition 2.8. *If **(H1)** holds, then for any $|\lambda|>\rho$ and $\omega\in\Omega_1$,*

$$\lim_{t\to\infty}\frac{\Xi_t}{t}\to-\gamma'(\lambda),\ \widetilde{\mathrm{P}}_x^{\xi,\lambda}\text{-a.s.}. \tag{2.34}$$

If we further assume that $\gamma(\lambda)>(m-1)\mathrm{es}$, then there exist Ω_λ with $\mathbb{P}(\Omega_\lambda)=1$ and a constant $\Sigma_\lambda^2>0$ such that for $\omega\in\Omega_\lambda$, when $\lambda>\rho$,

$$\frac{T_{-k}-\widetilde{\mathrm{E}}_x^{\xi,\lambda}T_{-k}}{\sqrt{k}} \overset{\mathrm{d}}{\Rightarrow} \mathcal{N}\left(0,\Sigma_\lambda^2\right) \quad \textit{under } \widetilde{\mathrm{P}}_x^{\xi,\lambda};$$

and when $\lambda<-\rho$,

$$\frac{T_{k}-\widetilde{\mathrm{E}}_x^{\xi,\lambda}T_{k}}{\sqrt{k}} \overset{\mathrm{d}}{\Rightarrow} \mathcal{N}\left(0,\Sigma_\lambda^2\right) \quad \textit{under } \widetilde{\mathrm{P}}_x^{\xi,\lambda}.$$

Proof. The idea of the proof is from [18, Lemma 2.6 and Corollary 2.7]. (i) Fix λ,x and ω. For any $\varepsilon>0$, by the definition of $\gamma'(\lambda)$, there exists $\delta:=\delta(\varepsilon,\lambda,\rho)\in(0,|\lambda|-\rho)$ such that for any $0<|\eta|<\delta$,

$$\left|\frac{\gamma(\lambda+\eta)-\gamma(\lambda)}{\eta}-\gamma'(\lambda)\right|<\varepsilon.$$

Our first goal is to show that

$$\lim_{t\to\infty}\frac{1}{t}\ln\widetilde{\mathrm{E}}_x^{\xi,\lambda}\left(e^{\eta\Xi_t}\right)=\gamma(\lambda-\eta)-\gamma(\lambda). \tag{2.35}$$

By (2.2) and (2.3),

$$\begin{aligned}
\frac{1}{t}\ln\widetilde{\mathrm{E}}_x^{\xi,\lambda}\left(e^{\eta\Xi_t}\right) &= \frac{1}{t}\ln\Pi_x\left(\exp\left\{\eta B_t-\gamma(\lambda)t+\int_0^t(m-1)\xi(B_s)\mathrm{d}s-\lambda(B_t-x)\right\}\frac{\phi(B_t,\lambda)}{\phi(x,\lambda)}\right)\\
&= -\gamma(\lambda)-\frac{-x\lambda+\ln\phi(x,\lambda)}{t}+\frac{1}{t}\ln\Pi_x\left(\exp\left\{(\eta-\lambda)B_t+\int_0^t(m-1)\xi(B_s)\mathrm{d}s\right\}\phi(B_t,\lambda)\right)\\
&= -\gamma(\lambda)-\frac{-x\lambda+\ln\phi(x,\lambda)}{t}
\end{aligned}$$

$$+\frac{1}{t}\ln\Pi_x\left(\exp\left\{(\eta-\lambda)B_t+\int_0^t(m-1)\xi(B_s)\mathrm{d}s\right\}\phi(B_t,\lambda-\eta)\frac{\phi(B_t,\lambda)}{\phi(B_t,\lambda-\eta)}\right). \quad (2.36)$$

Now fix η. Since $\ln\phi(y,\lambda)/y$ and $\ln\phi(y,\lambda-\eta)/y$ converge to 0 as $|y|\to\infty$, for any $\epsilon>0$, there exists an $M_1=M_1(\epsilon,\delta,\lambda)$ such that for all $y\in\mathbb{R}$,

$$M_1^{-1}e^{-\epsilon|y|}\le\phi(y,\lambda)\le M_1e^{\epsilon|y|},\quad M_1^{-1}e^{-\epsilon|y|}\le\phi(y,\lambda-\eta)\le M_1e^{\epsilon|y|}.$$

Thus, for all $y\in\mathbb{R}$,

$$M_1^{-2}e^{-2\epsilon|y|}\le\frac{\phi(y,\lambda)}{\phi(y,\lambda-\eta)}\le M_1^2e^{2\epsilon|y|}.$$

Applying the above to (2.36), we get

$$\begin{aligned}&\frac{1}{t}\ln\widetilde{\mathrm{E}}_x^{\xi,\lambda}\left(e^{\eta\Xi_t}\right)\\&\ge-\gamma(\lambda)+\frac{1}{t}\ln\Pi_x\left(e^{(\eta-\lambda)B_t-2\epsilon|B_t|+\int_0^t(m-1)\xi(B_s)\mathrm{d}s}\phi(B_t,\lambda-\eta)\right)+O\left(\frac{1}{t}\right)\\&=-\gamma(\lambda)+\gamma(\lambda-\eta)+\frac{1}{t}\ln\widetilde{\mathrm{E}}_x^{\xi,\lambda-\eta}\left(e^{-2\epsilon|\Xi_t|}\right)+O\left(\frac{1}{t}\right),\end{aligned} \quad (2.37)$$

here $O(1/t)=\left(\ln(M_1^{-2})-(-x\lambda+\ln\phi(x,\lambda)-\ln\phi(x,\lambda-\eta))\right)/t$. Similarly

$$\frac{1}{t}\ln\widetilde{\mathrm{E}}_x^{\xi,\lambda}\left(e^{\eta\Xi_t}\right)\le-\gamma(\lambda)+\gamma(\lambda-\eta)+\frac{1}{t}\ln\widetilde{\mathrm{E}}_x^{\xi,\lambda-\eta}\left(e^{2\epsilon|\Xi_t|}\right)+O\left(\frac{1}{t}\right) \quad (2.38)$$

with $O(1/t)=\left(\ln(M_1^2)-(-x\lambda+\ln\phi(x,\lambda)-\ln\phi(x,\lambda-\eta))\right)/t$. Note under $\widetilde{\mathrm{P}}_x^{\xi,\lambda-\eta}$, Ξ_t satisfies

$$\Xi_t=W_t+\int_0^t\frac{\psi_x(\Xi_s,\lambda-\eta)}{\psi(\Xi_s,\lambda-\eta)}\mathrm{d}s.$$

Since $\phi(\cdot,\lambda-\eta)\in\mathcal{A}_\infty$, $\phi_x(\cdot,\lambda-\eta)/\phi(\cdot,\lambda-\eta)\in L^\infty(\mathbb{R})$, by (2.1), there exists a constant $M_{\lambda-\eta}$, depending only on $\lambda-\eta$, such that for all $y\in\mathbb{R}$,

$$\left|\frac{\psi_x(y,\lambda-\eta)}{\psi(y,\lambda-\eta)}\right|=\left|-(\lambda-\eta)+\frac{\phi_x(y,\lambda-\eta)}{\phi(y,\lambda-\eta)}\right|\le M_{\lambda-\eta}. \quad (2.39)$$

Therefore, under $\widetilde{\mathrm{P}}_x^{\xi,\lambda-\eta}$, $|\Xi_t|\le|W_t-x|+|x|+M_{\lambda-\eta}t$. Since $\left(W_t-x,\widetilde{\mathrm{P}}_x^{\xi,\lambda-\eta}\right)\overset{\mathrm{d}}{=}(B_t,\Pi_0)$, by (2.37), we have

$$\begin{aligned}&\liminf_{t\to\infty}\frac{1}{t}\ln\widetilde{\mathrm{E}}_x^{\xi,\lambda}\left(e^{\eta\Xi_t}\right)\\&\ge-\gamma(\lambda)+\gamma(\lambda-\eta)+\liminf_{t\to\infty}\frac{1}{t}\ln\widetilde{\mathrm{E}}_x^{\xi,\lambda-\eta}\left(e^{-2\epsilon|\Xi_t|}\right)\\&\ge-\gamma(\lambda)+\gamma(\lambda-\eta)-2\epsilon M_{\lambda-\eta}+\liminf_{t\to\infty}\frac{1}{t}\ln\Pi_0\left(e^{-2\epsilon|B_t|}\right).\end{aligned} \quad (2.40)$$

For fixed ϵ, when t is large enough,

$$\Pi_0\left(e^{-2\epsilon|B_t|}\right)=\sqrt{\frac{2}{\pi}}\int_0^\infty e^{-2\epsilon\sqrt{t}y-y^2/2}\mathrm{d}y=e^{-2\epsilon^2t}\sqrt{\frac{2}{\pi}}\int_{2\epsilon\sqrt{t}}^\infty e^{-y^2/2}\mathrm{d}y\ge e^{-8\epsilon^2t}, \quad (2.41)$$

where in the last inequality we used the fact that

$$\left(\frac{1}{z}-\frac{1}{z^3}\right)e^{-z^2/2}\le\int_z^\infty e^{-y^2/2}\mathrm{d}y,\quad z>0$$

and the inequality $\sqrt{2/\pi}(1/z - 1/z^3) \geq e^{-z^2}$ for $z > 0$ large. Combining (2.40) and (2.41), we get

$$\liminf_{t\to\infty} \frac{1}{t} \ln \widetilde{\mathrm{E}}_x^{\xi,\lambda} \left(e^{\eta \Xi_t}\right) \geq -\gamma(\lambda) + \gamma(\lambda - \eta) - 2\epsilon M_{\lambda-\eta} - 8\epsilon^2. \tag{2.42}$$

Similarly,

$$\Pi_0 \left(e^{2\epsilon |B_t|}\right) = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{2\epsilon\sqrt{t}y - y^2/2} \mathrm{d}y \leq \sqrt{\frac{2}{\pi}} \int_{-\infty}^\infty e^{2\epsilon\sqrt{t}y - y^2/2} \mathrm{d}y = \sqrt{\frac{2}{\pi}} e^{2\epsilon^2 t}. \tag{2.43}$$

Using (2.38) and (2.43), we get

$$\limsup_{t\to\infty} \frac{1}{t} \ln \widetilde{\mathrm{E}}_x^{\xi,\lambda} \left(e^{\eta \Xi_t}\right) \leq -\gamma(\lambda) + \gamma(\lambda - \eta) + 2\epsilon M_{\lambda-\eta} + 2\epsilon^2. \tag{2.44}$$

Since $M_{\lambda-\eta}$ is independent of ϵ, let $\epsilon \to 0+$ in (2.42) and (2.44), we get (2.35).

Our second goal is to show that, for any $\epsilon_1 > 0$, there exist $M_3 = M_3(\lambda, \epsilon_1) > 0$ and $\delta_1 = \delta_1(\epsilon_1, \lambda) > 0$ such that for all t,

$$\widetilde{\mathrm{P}}_x^{\xi,\lambda} \left(|\Xi_t + \gamma'(\lambda)t| > \epsilon_1 t\right) \leq M_3 e^{-\delta_1 t}. \tag{2.45}$$

For any $\epsilon_1 > 0$, let $\varepsilon = \epsilon_1/4$ and let δ satisfy the condition at the beginning of the proof. By (2.35), for fixed $\eta \in (0, \delta)$, there exists $M_4 = M_4(\eta, \lambda, \epsilon_1)$ such that for all t,

$$\begin{aligned} \ln \widetilde{\mathrm{E}}_x^{\xi,\lambda} \left(e^{-\eta \Xi_t}\right) &\leq M_4 + (\gamma(\lambda + \eta) - \gamma(\lambda) + \epsilon_1 \eta/4)\, t \leq M_4 + \gamma'(\lambda)\eta t + \epsilon_1 \eta t/2, \\ \ln \widetilde{\mathrm{E}}_x^{\xi,\lambda} \left(e^{\eta \Xi_t}\right) &\leq M_4 + (\gamma(\lambda - \eta) - \gamma(\lambda) + \epsilon_1 \eta/4)\, t \leq M_4 - \gamma'(\lambda)\eta t + \epsilon_1 \eta t/2. \end{aligned}$$

Thus, by Markov's inequality,

$$\begin{aligned} &\widetilde{\mathrm{P}}_x^{\xi,\lambda} \left(\Xi_t < (-\gamma'(\lambda) - \epsilon_1)\, t\right) \leq \exp\left\{\ln \widetilde{\mathrm{E}}_x^{\xi,\lambda} \left(e^{-\eta \Xi_t}\right) - (\gamma'(\lambda) + \epsilon_1)\, \eta t\right\} \leq \exp\left\{M_4 - \eta \epsilon_1 t/2\right\}, \\ &\widetilde{\mathrm{P}}_x^{\xi,\lambda} \left(\Xi_t > (-\gamma'(\lambda) + \epsilon_1)\, t\right) \leq \exp\left\{\ln \widetilde{\mathrm{E}}_x^{\xi,\lambda} \left(e^{\eta \Xi_t}\right) + (\gamma'(\lambda) - \epsilon_1)\, \eta t\right\} \leq \exp\left\{M_4 - \eta \epsilon_1 t/2\right\}. \end{aligned}$$

Taking $M_3 = 2e^{M_4}$ and $\delta_1 = \eta\epsilon_1/2$, we obtain (2.45).

Finally, for any $\epsilon_1 > 0$ and $n > 2\left(M_\lambda + |\gamma'(\lambda)|\right)/\epsilon_1$ where M_λ is defined in (2.39), by Markov's inequality, we have

$$\begin{aligned} &\widetilde{\mathrm{P}}_x^{\xi,\lambda} \left(\sup_{n \leq t \leq n+1} |\Xi_t - \Xi_n + \gamma'(\lambda)(t-n)| > \epsilon_1 n\right) \\ &= \widetilde{\mathrm{P}}_x^{\xi,\lambda} \left(\sup_{n \leq t \leq n+1} \left|W_t - W_n + \int_n^t \psi_x(\Xi_s, \lambda)/\psi(\Xi_s, \lambda) \mathrm{d}s + \gamma'(\lambda)(t-n)\right| > \epsilon_1 n\right) \\ &\leq \Pi_0 \left(\sup_{t \in [0,1]} |B_t| > \epsilon_1 n - M_\lambda - |\gamma'(\lambda)|\right) \leq \Pi_0 \left(\sup_{t \in [0,1]} |B_t| > \epsilon_1 n/2\right) \\ &= 2\Pi_0(B_1 > \epsilon_1 n/2) \leq 2 \frac{\Pi_0\left(e^{\epsilon_1 n B_1/4}\right)}{e^{\epsilon_1^2 n^2/8}} = 2e^{-\epsilon_1^2 n^2/16}, \end{aligned} \tag{2.46}$$

where in the first inequality we used (2.39). Together with (2.45) and (2.46), we conclude that

$$\begin{aligned} &\sum_{n=1}^\infty \widetilde{\mathrm{P}}_x^{\xi,\lambda} \left(\sup_{n \leq t \leq n+1} |\Xi_t + \gamma'(\lambda)t| > 2\epsilon_1 n\right) \\ &\leq \sum_{n=1}^\infty \widetilde{\mathrm{P}}_x^{\xi,\lambda} \left(|\Xi_n + \gamma'(\lambda)n| > \epsilon_1 n\right) + \sum_{n=1}^\infty \widetilde{\mathrm{P}}_x^{\xi,\lambda} \left(\sup_{n \leq t \leq n+1} |\Xi_t - \Xi_n + \gamma'(\lambda)(t-n)| > \epsilon_1 n\right) < \infty. \end{aligned}$$

Now (2.34) follows from the Borel-Cantelli lemma.

(ii) Assume $\lambda > \rho$. For $i \in \mathbb{N}$, define

$$F_2(x,-i,\omega) := \widetilde{\mathrm{E}}_x^{\xi,\lambda}\left(T_{-i} - T_{-i+1} - \widetilde{\mathrm{E}}_x^{\xi,\lambda})T_{-i} - T_{-i+1})\right)^2,$$
$$F_4(x,-i,\omega) := \widetilde{\mathrm{E}}_x^{\xi,\lambda}\left(T_{-i} - T_{-i+1} - \widetilde{\mathrm{E}}_x^{\xi,\lambda}(T_{-i} - T_{-i+1})\right)^4.$$

We claim that $\mathbb{P}$-almost surely for all $i \geq 1,\ F_2(x,i,\omega) = F_2(x,1,\theta_{-i+1}\omega)$ and $F_4(x,i,\omega) = F_4(x,1,\theta_{-i+1}\omega)$. By the binomial theorem, to prove the claim above, it suffices to show that, for any $m = 1,\dots,4$ and $i \in \mathbb{N}$, $\mathrm{E}_x(T_{-i} - T_{-i+1})^m$ is stationary. By (2.2) and the strong Markov property of Ξ_t,

$$\begin{aligned}
&\widetilde{\mathrm{E}}_x^{\xi,\lambda}\left(T_{-i} - T_{-i+1}\right)^m = \widetilde{\mathrm{E}}_{-i+1+x}^{\xi,\lambda}\left(T_{-1}^m\right)\\
&= \Pi_{-i+1+x}\left(e^{-\gamma(\lambda)H_{-i+x} + \int_0^{H_{-i+x}}(m-1)\xi(B_s,\omega)\mathrm{d}s}\frac{\psi(-i+x,\lambda,\omega)}{\psi(-i+1+x,\lambda,\omega)}H_{-i+x}^m\right)\\
&= \Pi_x\left(e^{-\gamma(\lambda)H_{-1+x} + \int_0^{H_{-1+x}}(m-1)\xi(B_s,\theta_{-i+1}\omega)\mathrm{d}s}\frac{\psi(-1+x,\lambda,\theta_{-i+1}\omega)}{\psi(x,\lambda,\theta_{-i+1}\omega)}H_{-1+x}^m\right),
\end{aligned}$$

here in the last equality, we used the fact that

$$\frac{\psi(-i+x,\lambda,\omega)}{\psi(-i+1+x,\lambda,\omega)} = e^{\lambda}\phi(-1,\lambda,\theta_{-i+1+x}\omega) = \frac{\psi(-1+x,\lambda,\theta_{-i+1}\omega)}{\psi(x,\lambda,\theta_{-i+1}\omega)},$$

which is true by (2.11). Thus $\mathrm{E}_x(T_{-i} - T_{-i+1})^m$ is stationary. Note that we have assumed $\lambda > \rho$ and $\gamma(\lambda) > (m-1)\mathrm{es}$, so by Jensen's inequality, the trivial inequality $\mathbb{E}(X - \mathbb{E}X)^4 \leq 16\mathbb{E}(X^4)$ and (2.2),

$$\begin{aligned}
0 &< (\mathbb{E}F_2(x,-1,\omega))^2 \leq \mathbb{E}F_2^2(x,-1,\omega) \leq \mathbb{E}F_4(x,-1,\omega) \leq 16\mathbb{E}\widetilde{\mathrm{E}}_x^{\lambda}T_{-1}^4\\
&= 16\mathbb{E}\left\{\frac{\Pi_x\left(\exp\left\{\int_0^{H_{x-1}}\left((m-1)\xi(B_s,\omega) - \gamma(\lambda)\right)\mathrm{d}s\right\}H_{x-1}^4\right)}{\Pi_x\left(\exp\left\{\int_0^{H_{x-1}}\left((m-1)\xi(B_s,\omega) - \gamma(\lambda)\right)\mathrm{d}s\right\}\right)}\right\}\\
&\leq 16\frac{\Pi_x\left(\exp\left\{-\left(\gamma(\lambda) - (m-1)\mathrm{es}\right)H_{x-1}\right\}H_{x-1}^4\right)}{\Pi_x\left(\exp\left\{-\left(\gamma(\lambda) - (m-1)\mathrm{ei}\right)H_{x-1}\right\}\right)} < \infty.
\end{aligned}$$

Applying Birkhoff's ergodic theorem, we get that $\mathbb{P}$-almost surely, both $\frac{1}{k}\sum_{i=1}^k F_2(x,-i,\omega)$ and $\frac{1}{k}\sum_{i=1}^k F_4(x,-i,\omega)$ converge as $k \to \infty$. Let

$$\begin{aligned}
\Omega_\lambda := \Omega_1 \bigcap &\left\{\omega : \lim_{k\to+\infty}\frac{\sum_{i=1}^k F_2(x,-i,\omega)}{k} = \mathbb{E}F_2(x,-1,\omega)\right\}\\
\bigcap &\left\{\omega : \lim_{k\to+\infty}\frac{\sum_{i=1}^k F_4(x,-i,\omega)}{k} = \mathbb{E}F_4(x,-1,\omega)\right\}. \qquad (2.47)
\end{aligned}$$

Then $\mathbb{P}(\Omega_\lambda) = 1$ and for $\omega \in \Omega_\lambda$,

$$\lim_{k\to\infty}\frac{1}{\left(\sum_{i=1}^k F_2(x,-i,\omega)\right)^2}\sum_{i=1}^k F_4(x,-i,\omega) = 0,$$

which implies by Lyapunov's theorem that $(T_{-k} - \widetilde{\mathrm{E}}_x^{\xi,\lambda}T_{-k})/\sqrt{k}$ converges in $\widetilde{\mathrm{P}}_x^{\xi,\lambda}$- distribution to $\mathcal{N}\left(0,\mathbb{E}F_2(x,-1,\omega)\right)$. Setting $\Sigma_\lambda^2 := \mathbb{E}F_2(x,-1,\omega)$, we get the desired conclusion. □

Remark 2.9. By the strong Markov property of Ξ, $\{T_{-i} - T_{-i+1}, i \geq 1\}$ are independent (see (4.2) below). For any $M > 0$ and any subsequence $\{T_{-n_k}\}$ of $\{T_{-k}\}$, $\{\liminf_{k\to\infty}(T_{-n_k} - \widetilde{\mathrm{E}}_x^{\xi,\lambda} T_{-n_k})/\sqrt{n_k} < -M\}$ belongs to the tail σ-field of $\{T_{-i} - T_{-i+1}, i \geq 1\}$. By the central limit theorem, we have

$$\widetilde{\mathrm{P}}_x^{\xi,\lambda}\left(\liminf_{t\to\infty}\frac{T_{-n_m} - \widetilde{\mathrm{E}}_x^{\xi,\lambda}T_{-n_m}}{\sqrt{n_m}} < -M\right) = \lim_{k\to\infty}\widetilde{\mathrm{P}}_x^{\xi,\lambda}\left(\inf_{m\geq k}\frac{T_{-n_m} - \widetilde{\mathrm{E}}_x^{\xi,\lambda}T_{-n_m}}{\sqrt{n_m}} < -M\right)$$
$$\geq \lim_{k\to\infty}\widetilde{\mathrm{P}}_x^{\xi,\lambda}\left(\frac{T_{-n_k} - \widetilde{\mathrm{E}}_x^{\xi,\lambda}T_{-n_k}}{\sqrt{n_k}} < -M\right) > 0.$$

Thus by Kolmogorov's 0-1 law,

$$\widetilde{\mathrm{P}}_x^{\xi,\lambda}\left(\liminf_{k\to\infty}\frac{T_{-n_k} - \widetilde{\mathrm{E}}_x^{\xi,\lambda}T_{-n_k}}{\sqrt{n_k}} < -M\right) = 1.$$

Letting $M \to +\infty$, we get

$$\widetilde{\mathrm{P}}_x^{\xi,\lambda}\left(\liminf_{k\to\infty}\frac{T_{-n_k} - \widetilde{\mathrm{E}}_x^{\xi,\lambda}T_{-n_k}}{\sqrt{n_k}} = -\infty\right) = 1.$$

3 Proof of the main results

3.1 Proof of Theorem 1.4

Proof of Theorem 1.4. We only consider the case $\lambda > \rho$. Define

$$\widetilde{W}_t(\lambda) := e^{-\gamma(\lambda)t}\sum_{\nu\in N(t)} e^{-\lambda \widetilde{X}_t^\nu}\phi(\widetilde{X}_t^\nu,\lambda) = e^{-\gamma(\lambda)t}\sum_{\nu\in N(t)}\psi(\widetilde{X}_t^\nu,\lambda)$$

and $\widetilde{W}_\infty(\lambda) := \limsup_{t\to\infty}\widetilde{W}_t(\lambda)$. By [11, Theorem 4.3.5. p.227],

$$\widetilde{W}_\infty(\lambda) = +\infty,\ \widetilde{\mathrm{P}}_x^{\xi,\lambda}\text{-a.s.} \iff W_\infty(\lambda) = 0,\ \mathrm{P}_x^\xi\text{-a.s.};$$
$$\widetilde{W}_\infty(\lambda) < +\infty,\ \widetilde{\mathrm{P}}_x^{\xi,\lambda}\text{-a.s.} \iff \mathrm{E}_x^\xi W_\infty(\lambda) = e^{-\lambda x}\phi(x,\lambda).$$

Define

$$\Omega_2 := \Omega_{\lambda^*}\cap\Omega_{-\lambda^*}\bigcap\left\{\limsup_{n\to\infty}\frac{\ln\psi(-n+x,\lambda^*) - \lambda^*\frac{\psi_\lambda(-n+x,\lambda^*)}{\psi(-n+x,\lambda^*)}}{\sqrt{n}} \geq 0\right\}$$
$$\bigcap\left\{\limsup_{n\to\infty}\frac{\ln\psi(n+x,-\lambda^*) - \lambda^*\frac{\psi_\lambda(n+x,-\lambda^*)}{\psi(n+x,-\lambda^*)}}{\sqrt{n}} \geq 0\right\},$$

here Ω_{λ^*} and $\Omega_{-\lambda^*}$ are defined in (2.47). Then $\mathbb{P}(\Omega_2) = 1$ by **(H1)**, (2.22) and Lemma 2.6. We will fix $\omega\in\Omega_2$ in the rest of the proof.

(i) We first consider the case $\lambda > \lambda^*$. In this case,

$$\widetilde{W}_t(\lambda) \geq e^{-\gamma(\lambda)t}\psi(\Xi_t,\lambda) = \exp\left\{\left(\frac{\ln\psi(\Xi_t,\lambda)}{\Xi_t}\times\frac{\Xi_t}{t} - \gamma(\lambda)\right)t\right\}.$$

By Lemma 2.8 and the asymptotic behavior of $\ln\psi(\cdot,\lambda)$, we have

$$\lim_{t\to\infty}\frac{\ln\psi(\Xi_t,\lambda)}{\Xi_t}\frac{\Xi_t}{t} = (-\lambda)\cdot(-\gamma'(\lambda)) = \lambda\gamma'(\lambda),\ \widetilde{\mathrm{P}}_x^{\xi,\lambda}\text{-a.s.}.$$

By Lemma 2.1, when $|\lambda| > \lambda^*$, $\lambda\gamma'(\lambda) > \gamma(\lambda)$. Thus, $\widetilde{W}_\infty(\lambda) = +\infty$, $\widetilde{\mathrm{P}}_x^{\xi,\lambda}$-a.s. or equivalently, $W_\infty(\lambda) = 0$, P_x^ξ -a.s..

Now we consider the case $\lambda = \lambda^*$. In this case, we first prove that for any $M > 0$,

$$\widetilde{\mathrm{P}}_x^{\xi,\lambda^*}\left(\ln\psi(\Xi_t,\lambda^*) - \gamma(\lambda^*)t > M \text{ i.o.}\right) = 1.$$

Note that by (2.2),

$$\gamma(\lambda^*)\widetilde{\mathrm{E}}_x^{\xi,\lambda} T_{-n} = \gamma(\lambda^*)\Pi_x\left(e^{\int_0^{H_{-n+x}}((m-1)\xi(B_s)-\gamma(\lambda^*))\mathrm{d}s} H_{-n+x}\right)\frac{\psi(-n+x,\lambda^*)}{\psi(x,\lambda^*)}.$$

By Lemma 2.1, $\gamma(\lambda^*) = \lambda^*\gamma'(\lambda^*)$. Thus, using (2.4) and (2.14), we have

$$\begin{aligned}\gamma(\lambda^*)\widetilde{\mathrm{E}}_x^{\xi,\lambda} T_{-n} &= \lambda^*\gamma'(\lambda^*)\frac{\Pi_x\left(\exp\left\{\int_0^{H_{-n+x}}((m-1)\xi(B_s)-\gamma(\lambda^*))\,\mathrm{d}s\right\}H_{-n+x}\right)}{\Pi_x\left(\exp\left\{\int_0^{H_{-n+x}}((m-1)\xi(B_s)-\gamma(\lambda^*))\,\mathrm{d}s\right\}\right)}\\ &= \lambda^*\left(\frac{\psi_\lambda(-n+x,\lambda^*)}{\psi(-n+x,\lambda^*)} - \frac{\psi_\lambda(x,\lambda^*)}{\psi(x,\lambda^*)}\right).\end{aligned}$$

Therefore,

$$\begin{aligned}&\widetilde{\mathrm{P}}_x^{\xi,\lambda^*}\left(\ln\psi(\Xi_t,\lambda^*) - \gamma(\lambda^*)t > M \text{ i.o.}\right) \geq \widetilde{\mathrm{P}}_x^{\xi,\lambda^*}\left(\ln\psi(\Xi_{T_{-n}},\lambda^*) - \gamma(\lambda^*)T_{-n} > M \text{ i.o.}\right)\\ &= \widetilde{\mathrm{P}}_x^{\xi,\lambda^*}\left(\gamma(\lambda^*)\frac{T_{-n} - \widetilde{\mathrm{E}}_x^{\xi,\lambda^*}T_{-n}}{\sqrt{n}} < \frac{\ln\psi(-n+x,\lambda) - \gamma(\lambda^*)\widetilde{\mathrm{E}}_x^{\xi,\lambda^*}T_{-n} - M}{\sqrt{n}} \text{ i.o.}\right)\\ &= \widetilde{\mathrm{P}}_x^{\xi,\lambda^*}\Bigg(\gamma(\lambda^*)\frac{T_{-n} - \widetilde{\mathrm{E}}_x^{\xi,\lambda^*}T_{-n}}{\sqrt{n}}\\ &\qquad < \frac{\ln\psi(-n+x,\lambda^*) - \lambda^*\left(\frac{\psi_{\lambda^*}(-n+x,\lambda^*)}{\psi(-n+x,\lambda^*)} - \frac{\psi_{\lambda^*}(x,\lambda^*)}{\psi(x,\lambda^*)}\right) - M}{\sqrt{n}} \text{ i.o.}\Bigg).\end{aligned}$$

Thanks to our choice of Ω_2, there exists a subsequence $\{n_k = n_k(\omega), k \geq 1\}$ such that for all $\omega \in \Omega_2$,

$$\lim_{k\to\infty}\frac{\ln\psi(-n_k+x,\lambda^*) - \lambda^*\left(\frac{\psi_{\lambda^*}(-n_k+x,\lambda^*)}{\psi(-n_k+x,\lambda^*)} - \frac{\psi_{\lambda^*}(x,\lambda^*)}{\psi(x,\lambda^*)}\right) - M}{\sqrt{n_k}} \geq 0,$$

so

$$\inf_{k\geq 1}\frac{\ln\psi(-n_k+x,\lambda^*) - \lambda^*\left(\frac{\psi_{\lambda^*}(-n_k+x,\lambda^*)}{\psi(-n_k+x,\lambda^*)} - \frac{\psi_{\lambda^*}(x,\lambda^*)}{\psi(x,\lambda^*)}\right) - M}{\sqrt{n_k}} > -\infty.$$

Together with Remark 2.9, we have

$$\begin{aligned}&\widetilde{\mathrm{P}}_x^{\xi,\lambda^*}\left(\ln\psi(\Xi_t,\lambda^*) - \gamma(\lambda^*)t > M \text{ i.o.}\right)\\ &\geq \widetilde{\mathrm{P}}_x^{\xi,\lambda^*}\Bigg(\gamma(\lambda^*)\frac{T_{-n} - \widetilde{\mathrm{E}}_x^{\xi,\lambda^*}T_{-n}}{\sqrt{n}}\\ &\qquad < \frac{\ln\psi(-n+x,\lambda^*) - \lambda^*\left(\frac{\psi_{\lambda^*}(-n+x,\lambda^*)}{\psi(-n+x,\lambda^*)} - \frac{\psi_{\lambda^*}(x,\lambda^*)}{\psi(x,\lambda^*)}\right) - M}{\sqrt{n}} \text{ i.o.}\Bigg)\\ &= 1.\end{aligned}$$

Letting $M \to \infty$,

$$\widetilde{\mathrm{P}}_x^{\xi,\lambda^*}\left(\limsup_{t\to\infty}\left(\ln\psi(\Xi_t,\lambda^*) - \gamma(\lambda^*)t\right) = +\infty\right) = 1,$$

this implies $\widetilde{W}_\infty(\lambda^*) = +\infty,\ \widetilde{\mathrm{P}}_x^{\xi,\lambda^*}$-a.s. or equivalently, $W_\infty(\lambda^*) = 0,\ \mathrm{P}_x^{\xi}$-a.s..

(ii) When $|\lambda| \in (\rho, \lambda^*)$, by Lemma 2.1, $\lambda\gamma'(\lambda) - \gamma(\lambda) < 0$. Let O be the law of the offspring of the spine particle, that is, $\widetilde{\mathrm{P}}_x^{\xi,\lambda}(O = k) = kp_k/m$ for $k = 1, 2, \ldots$. If $\sum_{k=1}^{\infty}(k \ln k)p_k = +\infty$, then

$$\infty = \widetilde{\mathrm{E}}_x^{\xi,\lambda}\left(\frac{\ln^+ O}{M}\right) = \int_0^\infty \widetilde{\mathrm{P}}_x^{\xi,\lambda}(\ln^+ O > yM)\mathrm{d}y \le \sum_{k=0}^{\infty} \widetilde{\mathrm{P}}_x^{\xi,\lambda}(\ln^+ O > kM), \quad M > 0.$$

Let μ_k be the time of the kth fission of the spine and O_k be the number of offspring of the spine at time μ_k. Then by the Borel-Cantelli lemma and the arbitrariness of M, we have

$$\limsup_{k\to\infty} \frac{\ln O_k}{k} = +\infty,\ \widetilde{\mathrm{P}}_x^{\xi,\lambda}\text{-a.s..}$$

Note that

$$\widetilde{W}_{\mu_k} \ge (O_k - 1)e^{-\gamma(\lambda)\mu_k}\psi(\Xi_{\mu_k}, \lambda) = \exp\left\{\left(\frac{\ln^+(O_k - 1)}{\mu_k} + \frac{\ln\psi(\Xi_{\mu_k}, \lambda)}{\mu_k} - \gamma(\lambda)\right)\mu_k\right\}.$$

Since $(m-1)\xi(x) \ge (m-1)\mathrm{ei}$, we have

$$\widetilde{\mathrm{P}}_x^{\xi,\lambda}(\mu_k \le t) = \widetilde{\mathrm{P}}_x^{\xi,\lambda}(N_t \ge k) \ge \widetilde{\mathrm{P}}_x^{\xi,\lambda}(\tilde{N}_t \ge k) = \widetilde{\mathrm{P}}_x^{\xi,\lambda}(\tilde{\mu}_k \le t),$$

where, given the trajectory of Ξ, N_t is a Poisson process with rate $(m-1)\xi(\Xi_t)$ and $\widetilde{N}_t$ is a Poisson process with rate $(m-1)\mathrm{ei}$, $\tilde{\mu}_k := \inf\{t : \widetilde{N}_t = k\}$. We use a coupling of $(N_t, \widetilde{N}_t)$ such that $N_t - \tilde{N}_t$ is a Poisson process with rate $(m-1)\left(\xi(\Xi_t) - \mathrm{ei}\right)$ and is independent of $\tilde{N}_t$. So $\mu_k \le \tilde{\mu}_k$. Since $k/\tilde{\mu}_k \to ((m-1)\mathrm{ei})$, $\widetilde{\mathrm{P}}_x^{\xi,\lambda}$-a.s., we have $\liminf_{k\to\infty} k/\mu_k \ge ((m-1)\mathrm{ei})$, $\widetilde{\mathrm{P}}_x^{\xi,\lambda}$-a.s.. Similarly we also have $\limsup_{k\to\infty} k/\mu_k \le ((m-1)\mathrm{es})$, $\widetilde{\mathrm{P}}_x^{\xi,\lambda}$-a.s.. Thus, $\mu_k \to \infty$ $\widetilde{\mathrm{P}}_x^{\xi,\lambda}$-a.s. and

$$\limsup_{k\to\infty} \widetilde{W}_{\mu_k} \ge \limsup_{k\to\infty} \exp\left\{\left(\frac{\ln^+(O_k - 1)}{k}\frac{k}{\mu_k} + \frac{\ln\psi(\Xi_{\mu_k}, \lambda)}{\mu_k} - \gamma(\lambda)\right)\mu_k\right\} = +\infty.$$

If $\sum_{k=1}^{\infty}(k \ln k)p_k < +\infty$, then for any $c > 0$,

$$\infty > \widetilde{\mathrm{E}}_x^{\xi,\lambda}\left(\frac{\ln^+ O}{c}\right) = \int_0^\infty \widetilde{\mathrm{P}}_x^{\xi,\lambda}(\ln^+ O > yc)\mathrm{d}y \ge \sum_{k=1}^{\infty} \widetilde{\mathrm{P}}_x^{\xi,\lambda}(\ln^+ O > kc).$$

Since c is arbitrary, we have

$$\limsup_{k\to\infty} \frac{\ln O_k}{k} = 0, \quad \widetilde{\mathrm{P}}_x^{\xi,\lambda}\text{-a.s..}$$

Define $\widetilde{\mathcal{G}}$ to be the σ-field of the genealogy along the spine, then we have

$$\widetilde{\mathrm{E}}_x^{\xi,\lambda}\left(\widetilde{W}_t(\lambda)\middle|\widetilde{\mathcal{G}}\right) = \psi(\Xi_t, \lambda)e^{-\gamma(\lambda)t} + \sum_{k=1}^{N_t - 1}(O_k - 1)\psi(\Xi_{\mu_k}, \lambda)e^{-\gamma(\lambda)\mu_k}.$$

Here O_k, μ_k and N_t have the same meaning as in the case $\sum_{k=1}^{\infty}(k \ln k)p_k = +\infty$. Then, $\widetilde{\mathrm{P}}_x^{\xi,\lambda}$-a.s.,

$$\limsup_{k\to\infty} \frac{\ln(O_k - 1)}{\mu_k} = \limsup_{k\to\infty} \frac{\ln(O_k - 1)}{k}\frac{k}{\mu_k} = 0.$$

Note that $\widetilde{\mathrm{P}}_x^{\xi,\lambda}$-a.s.,

$$\lim_{t\to\infty} \frac{\ln\psi(\Xi_t, \lambda)}{t} = \lim_{t\to\infty} \frac{\ln\psi(\Xi_t, \lambda)}{\Xi_t}\frac{\Xi_t}{t} = \lambda\gamma'(\lambda) < \gamma(\lambda),$$

and thus

$$\lim_{t\to\infty}\frac{\ln\psi(\Xi_t,\lambda)}{t}<\frac{\gamma(\lambda)+\lambda\gamma'(\lambda)}{2},$$

which implies

$$\limsup_{t\to\infty}\psi(\Xi_t,\lambda)\exp\left\{-\left(\gamma(\lambda)+\lambda\gamma'(\lambda)\right)t/2\right\}=0.$$

So there exists a $\widetilde{\mathrm{P}}_x^{\xi,\lambda}$-a.s. finite random variable η such that for all k and t,

$$\ln(O_k-1)\le\eta+\frac{\gamma(\lambda)-\lambda\gamma'(\lambda)}{4}\mu_k,\ \text{and}\ \psi(\Xi_t,\lambda)\exp\left\{-\left(\gamma(\lambda)+\lambda\gamma'(\lambda)\right)t/2\right\}\le\eta.$$

Therefore,

$$\begin{aligned}\widetilde{\mathrm{E}}_x^{\xi,\lambda}\left(\widetilde{W}_t(\lambda)\middle|\widetilde{\mathcal{G}}\right)&=\psi(\Xi_t,\lambda)e^{-\gamma(\lambda)t}+\sum_{k=1}^{N_t-1}(O_k-1)\psi(\Xi_{\mu_k},\lambda)e^{-\gamma(\lambda)\mu_k}\\&\le\eta\exp\left\{-\left(\gamma(\lambda)-\lambda\gamma'(\lambda)\right)t/2\right\}+\sum_{k=1}^{\infty}\eta e^{\eta}\exp\left\{\exp\left\{-\left(\gamma(\lambda)-\lambda\gamma'(\lambda)\right)\mu_k/4\right\}\right\}<\infty,\end{aligned}$$

which implies $\limsup_{t\to\infty}\widetilde{\mathrm{E}}_x^{\xi,\lambda}\left(\widetilde{W}_t(\lambda)\middle|\widetilde{\mathcal{G}}\right)<\infty,\ \widetilde{\mathrm{P}}_x^{\xi,\lambda}$-a.s. Fatou's lemma for conditional probability implies that $\liminf_{t\to\infty}\widetilde{W}_t(\lambda)<\infty$, $\widetilde{\mathrm{P}}_x^{\xi,\lambda}$-a.s. Since $\widetilde{W}_t(\lambda)^{-1}$ is a non-negative supermartingale under $\widetilde{\mathrm{P}}_x^{\xi,\lambda}$, we have $\lim_{t\to\infty}\widetilde{W}_t(\lambda)=\widetilde{W}_\infty(\lambda)<\infty$, $\widetilde{\mathrm{P}}_x^{\xi,\lambda}$-a.s. Thus we have shown that when $\lambda\in(\rho,\lambda^*)$ and $\sum_{k=1}^{\infty}(k\ln k)p_k<\infty$, $W_t(\lambda)$ converges P_x^{ξ}-a.s. and in L^1 to $W_\infty(\lambda)$. □

Proof of Corollary 1.5. For any $\epsilon>0$, there exists constant $c_\epsilon>0$ such that for all $x\in\mathbb{R}$,

$$\psi(x,-\lambda^*)\ge c_\epsilon e^{\lambda^*x-\epsilon|x|}.$$

Taking $\lambda=-\lambda^*$ and using Theorem 1.4, we have, P_x^{ξ}-a.s.,

$$0=W_\infty(-\lambda^*)\ge\limsup_{t\to\infty}\psi(M_t,-\lambda^*)e^{-\gamma(\lambda^*)t}\ge c_\epsilon\limsup_{t\to\infty}\exp\left\{\lambda^*M_t-\epsilon|M_t|-\gamma(\lambda^*)t\right\}\ge0.$$

Thus, P_x^{ξ}-a.s.,

$$\lim_{t\to\infty}\lambda^*M_t-\epsilon|M_t|-\gamma(\lambda^*)t=-\infty,$$

which implies that

$$\limsup_{t\to\infty}\frac{\lambda^*M_t-\epsilon|M_t|}{t}\le\gamma(\lambda^*),\quad\mathrm{P}_x^{\xi}\text{-a.s.}.\tag{3.1}$$

When $\lambda\in(\rho,\lambda^*)$, we will first show that for any $\omega\in\Omega_2$, $\widetilde{\mathrm{P}}_x^{\xi,\lambda}$ and P_x^{ξ} are mutually absolutely continuous. Define

$$q(x,\lambda,\omega):=\mathrm{P}_x^{\xi}\left(W_\infty(\lambda)=0\right).$$

We only need to prove that

$$q(x,\lambda,\omega)=0,\quad\forall x\in\mathbb{R}.\tag{3.2}$$

By the branching property, for $s<t$, we have

$$W_t(\lambda)=e^{\gamma(\lambda)s}\sum_{\nu\in N(s)}W_{t-s}^{\nu}(\lambda).$$

Here conditioned on $\widetilde{\mathcal{F}}_s$, $\{W_{t-s}^{\nu}(\lambda),\nu\in N(s)\}$ are independent. Letting $t\to\infty$, we get

$$q(x,\lambda,\omega)=\mathrm{E}_x^{\xi}\left(\prod_{\nu\in N(s)}q(X_s^{\nu},\lambda,\omega)\right).$$

By the branching property and the Markov property, $\left\{\prod_{\nu\in N(s)} q(X_s^\nu,\lambda,\omega)\right\}_{s\geq 0}$ is a bounded martingale under P_x^ξ. Since $W_\infty(\lambda)$ is the L^1 limit of $W_t(\lambda)$, $q(x,\lambda,\omega)<1$ for all $x\in\mathbb{R}$. By (1.3), $q(\cdot,\lambda,\omega)$ satisfies the equation

$$q_t(x,\lambda,\omega)=\frac{1}{2}q_{xx}(x,\lambda,\omega)+\xi(x)\left(\sum_{k=1}^\infty p_k q^k(x,\lambda,\omega)-q(x,\lambda,\omega)\right).$$

By the Feynman-Kac formula, we have

$$q(x,\lambda,\omega)=\Pi_x\left(q(B_t,\lambda,\omega)\exp\left\{\int_0^t \xi(B_s)\left(\sum_{k=1}^\infty p_k q^{k-1}(B_s,\lambda,\omega)-1\right)\mathrm{d}s\right\}\right).$$

Let $A(\varepsilon):=\{x: q(x,\lambda,\omega)\leq 1-\varepsilon\}$, then

$$\begin{aligned} q(x,\lambda,\omega)&\leq \Pi_x\left(\exp\left\{\int_0^t \xi(B_s)\left(\sum_{k=1}^\infty p_k q^{k-1}(B_s,\lambda,\omega)-1\right)\mathrm{d}s\right\}\right)\\ &\leq \Pi_x\left(\exp\left\{\int_0^t \mathrm{ei}(1-p_1)\left(q(B_s,\lambda,\omega)-1\right)\mathrm{d}s\right\}\right)\\ &\leq \Pi_x\left(\exp\left\{-\varepsilon\mathrm{ei}(1-p_1)\int_0^t 1_{A(\varepsilon)}(B_s)\mathrm{d}s\right\}\right), \end{aligned}\tag{3.3}$$

where in the second inequality, we used the domination $\left(\sum_{k=1}^\infty p_k q^{k-1}(B_s,\lambda,\omega)-1\right)\leq p_1+(\sum_{k=2}^\infty p_i)q(B_s,\lambda,\omega)-1=(p_1-1)+(1-p_1)q(B_s,\lambda,\omega)$. Let m be the Lebesgue measure in $\mathbb{R}$. Note that if $m\left(A(\varepsilon)\right)>0$, then as $t\to\infty$,

$$\int_0^t 1_{A(\varepsilon)}(B_s)\mathrm{d}s\to+\infty,\ \Pi_x\text{-a.s.}.$$

Since $q(x,\lambda,\omega)<1$, we can find $\varepsilon=\varepsilon(\omega)>0$ such that $m\left(A(\varepsilon)\right)>0$. Letting $t\to\infty$ in (3.3), we obtain (3.2), and thus $\widetilde{\mathrm{P}}_x^{\xi,\lambda}$ and P_x^ξ are mutually absolute continuous. Noticing that

$$\widetilde{\mathrm{P}}_x^{\xi,\lambda}\left(\liminf_{t\to\infty}\frac{\widetilde{M}_t}{t}\geq\gamma'(\lambda)\right)\geq\widetilde{\mathrm{P}}_x^{\xi,\lambda}\left(\liminf_{t\to\infty}\frac{\Xi_t}{t}\geq\gamma'(\lambda)\right)=1,$$

we have

$$\mathrm{P}_x^\xi\left(\liminf_{t\to\infty}\frac{M_t}{t}\geq\gamma'(\lambda)\right)=1.$$

Let $\lambda\uparrow\lambda^*$ we get that

$$\mathrm{P}_x^\xi\left(\liminf_{t\to\infty}\frac{M_t}{t}\geq v^*\right)=1.\tag{3.4}$$

(H2) implies that $\lambda^*>\rho$ and $\gamma(\lambda^*)>(m-1)\mathrm{es}$. By Lemma 2.3 (4), $\gamma'(\lambda)\uparrow\gamma'(\lambda^*)=v^*$ as $\lambda\uparrow\lambda^*$. In view of (3.4), we know that $M_t\to+\infty,\ \mathrm{P}_x^\xi$-a.s. and thus (3.1) can be rewritten as

$$\limsup_{t\to\infty}\frac{M_t}{t}\leq\frac{\gamma(\lambda^*)}{\lambda^*-\epsilon},\ \mathrm{P}_x^\xi\text{-a.s.}.$$

Letting $\epsilon\to 0+$ and using (3.4), we get the conclusion of the corollary. □

3.2 Proof of Theorem 1.6

In this subsection, unless explicitly mentioned otherwise, we always assume $\lambda=-\lambda^*$ and that **(H1)**, **(H2)** and **(H3)** hold. The proof of Theorem 1.6 is inspired by [7, Section

6] and [10, Section 4]. Since the proof is long and complicated, we divide it into several parts. In this subsection,

$$K_1 := K_1(-\lambda^*), \quad K_2 := K_2(-\lambda^*)$$

are the two constants in Lemma 2.3(2).

For $s \in [0,t]$ and $\Gamma_0 > 0$, define

$$\varphi_t(s) := \sup\left\{k \in \mathbb{Z} : \ s \geq \frac{1}{\lambda^* v^*} \ln \psi(k, -\lambda^*) - \Gamma_0 \ln t\right\}.$$

Later we will take $\Gamma_0 > 2C(\ell') + 1$ with $C(\ell')$ being defined in Lemma 3.2 (i) below. We claim that for any $\Gamma_0 > 0$, $\varphi_t(s)$ is càdlàg and non-decreasing as a function of $s \in [0,t]$. In fact, if $\varphi_t(s) = k$, then by definition,

$$s \in \left[\frac{1}{\lambda^* v^*} \ln \psi(k, -\lambda^*) - \Gamma_0 \ln t, \ \frac{1}{\lambda^* v^*} \ln \psi(k+1, -\lambda^*) - \Gamma_0 \ln t\right), \tag{3.5}$$

which implies that, for any $r \in \left(s, \frac{1}{\lambda^* v^*} \ln \psi(k+1, -\lambda^*) - \Gamma_0 \ln t\right)$, $\varphi_t(r) = k$. On the other hand, suppose that $s_n \in [0,t]$, $s_n \uparrow s$, and that $\varphi_t(s) = k$. If $s = \frac{1}{\lambda^* v^*} \ln \psi(k, -\lambda^*) - \Gamma_0 \ln t$, then $\varphi_t(s_n) \uparrow (k-1)$, if $s > \frac{1}{\lambda^* v^*} \ln \psi(k, -\lambda^*) - \Gamma_0 \ln t$, then for n large enough, we have $\frac{1}{\lambda^* v^*} \ln \psi(k, -\lambda^*) - \Gamma_0 \ln t < s_n < s$, which implies that $\varphi_t(s_n) = k$. Thus $\varphi_t(s)$ is a càdlàg function of $s \in [0,t]$. The monotonicity follows easily from Remark 2.4.

Define

$$\begin{aligned}\mathcal{L}_t := \#\Big\{\nu \in N(t) : & X_t^\nu \geq V_t + 2, \min_{r \in [t-1,t]} X_r^\nu \geq V_t - 3, \ X_s^\nu < \varphi_t(s), \ \forall s \in \left[H_{[V_t]}^\nu, t\right], \\ & H_k^\nu \geq \frac{1}{\lambda^* v^*} \ln \psi(k+1, -\lambda^*) - \Gamma_0 \ln t, \ \forall k = 1, 2, ..., [V_t]\Big\},\end{aligned} \tag{3.6}$$

here for any $y \in \mathbb{R}$ and $\nu \in N(t)$, H_y^ν is the first time that the particle ν hits y, i.e., $H_y^\nu = \inf\{t \geq 0 : X_t^\nu = y\}$. To prove Theorem 1.6, we will use the following inequality:

$$\mathrm{P}_y^\xi(\mathcal{L}_t > 0) \geq \frac{\left(\mathrm{E}_y^\xi(\mathcal{L}_t)\right)^2}{\mathrm{E}_y^\xi\left((\mathcal{L}_t)^2\right)}, \quad t > 0.$$

We will estimate the first moment of $\mathcal{L}_t$ from below and the second moment of $\mathcal{L}_t$ from above. The following lemma gives the estimate of the second moment of $\mathcal{L}_t$.

Lemma 3.1. *There exists a non-random constant ℓ_2 such that for t large enough,*

$$\sup_{y \in [-1,1]} \mathrm{E}_y^\xi\left((\mathcal{L}_t)^2\right) \leq t^{\ell_2}.$$

Proof. Note when $\nu \in N(t)$ makes a contribution to $\mathcal{L}_t$, we have $X_s^\nu < \varphi_t(s)$ for all $s \in [0,t]$. This is because that for $s \in \left[H_1^\nu, H_{[V_t]}^\nu\right)$, there exists $k \in \{2, ..., [V_t]\}$ such that $s \in \left[H_{k-1}^\nu, H_k^\nu\right)$. Therefore,

$$s \geq H_{k-1}^\nu \geq \frac{1}{\lambda^* v^*} \ln \psi(k, -\lambda^*) - \Gamma_0 \ln t,$$

which implies that $\varphi_t(s) \geq k$. But $s < H_k^\nu$ means $X_s^\nu < k$, so we get that $X_s^\nu < k \leq \varphi_t(s)$ for $s \in \left[H_1^\nu, H_{[V_t]}\right)$. If $s \in [0, H_1^\nu)$, then when t is large enough so that $0 \geq \frac{1}{\lambda^* v^*} \ln \psi(1, -\lambda^*) - \Gamma_0 \ln t$,

$$s \geq 0 \geq \frac{1}{\lambda^* v^*} \ln \psi(1, -\lambda^*) - \Gamma_0 \ln t,$$

thus $\varphi_t(s) \geq k$ holds for $k=1$. Thus for $s < H_1^\nu$, we have $X_s^\nu < 1 \leq \varphi_t(s)$. In conclusion,

$$\mathcal{L}_t \leq \#\left\{\nu \in N(t) : X_t^\nu \geq V_t + 2, X_s^\nu < \varphi_t(s),\ \forall s \in [0,t]\right\}.$$

By Proposition 1.2,

$$\begin{aligned}\mathrm{E}_y^\xi\left((\mathcal{L}_t)^2\right) \leq \mathrm{E}_y^\xi\left(\mathcal{L}_t\right) + (m_2 - m)\int_0^t \Pi_y\Big(& e^{\int_0^s (m-1)\xi(B_r)\mathrm{d}r}\xi(B_s)1_{\{B_r<\varphi_t(r),\ \forall 0\leq r\leq s\}} \\ &\times\left(\Pi_z\left(e^{\int_0^{t-s}(m-1)\xi(B_r)\mathrm{d}r}1_{\{B_r\leq\varphi_t(r+s),\ \forall 0\leq r\leq t-s, B_{t-s}\geq V_t+2\}}\right)\right)^2_{|z=B_s}\Big)\mathrm{d}s.\end{aligned}$$

Thus, for $z < \varphi_t(s)$, using the monotonicity of $\psi(\cdot, -\lambda^*)$ (see Remark 2.4) and (1.7), we have

$$\begin{aligned}&\Pi_z\left(\exp\left\{\int_0^{t-s}(m-1)\xi(B_r)\mathrm{d}r\right\}1_{\{B_r\leq\varphi_t(r+s),\ \forall 0\leq r\leq t-s, B_{t-s}\geq V_t+2\}}\right) \\ &\leq \Pi_z\left(\exp\left\{\int_0^{t-s}(m-1)\xi(B_r)\mathrm{d}r\right\}1_{\{B_{t-s}\geq V_t+2\}}\right) \\ &\leq \Pi_z\left(\exp\left\{\int_0^{t-s}(m-1)\xi(B_r)\mathrm{d}r\right\}\frac{\psi(B_{t-s},-\lambda^*)}{\psi(V_t+2,-\lambda^*)}\right) = \frac{\psi(z,-\lambda^*)e^{\gamma(\lambda^*)(t-s)}}{\psi(V_t+2,-\lambda^*)} \\ &\leq \frac{\psi\left(\varphi_t(s),-\lambda^*\right)e^{\gamma(\lambda^*)(t-s)}}{\psi(V_t,-\lambda^*)} = \psi\left(\varphi_t(s),-\lambda^*\right)e^{-\lambda^* v^* s},\end{aligned} \tag{3.7}$$

where in the last equality we used the fact that $e^{-\gamma(\lambda^*)t}\psi(V_t,-\lambda^*) = 1$ given by (1.13). Since (3.5) implies that for $s \in [0,t]$,

$$s \geq \frac{1}{\lambda^* v^*}\ln\psi(\varphi_t(s),-\lambda^*) - \Gamma_0 \ln t,$$

the last term of (3.7) is bounded from above by $t^{\Gamma_0\lambda^* v^*}$ for all $z < \varphi_t(s)$, $s \in [0,t]$. Similarly as in (3.7), for $s = 0$ we have

$$\begin{aligned}&\Pi_y\left(\exp\left\{\int_0^t(m-1)\xi(B_r)\mathrm{d}r\right\}1_{\{B_t\geq V_t+2\}}\right) \\ &\leq \Pi_y\left(\exp\left\{\int_0^t(m-1)\xi(B_r)\mathrm{d}r\right\}\frac{\psi(B_t,-\lambda^*)}{\psi(V_t+2,-\lambda^*)}\right) = \frac{\psi(y,-\lambda^*)e^{\gamma(\lambda^*)t}}{\psi(V_t+2,-\lambda^*)} \\ &\leq \frac{\psi(1,-\lambda^*)e^{\gamma(\lambda^*)t}}{\psi(V_t,-\lambda^*)} = \psi(1,-\lambda^*) \leq e^{K_1},\end{aligned}$$

where in the last inequality we used (2.19). Therefore we conclude that

$$\begin{aligned}&\mathrm{E}_y^\xi\left((\mathcal{L}_t)^2\right) \\ &\leq \mathrm{E}_y^\xi\left(\mathcal{L}_t\right) + (m_2-m)\int_0^t\Pi_y\Big(\exp\left\{\int_0^s(m-1)\xi(B_r)\mathrm{d}r\right\}\xi(B_s)1_{\{B_r<\varphi_t(r),\ \forall 0\leq r\leq s\}} \\ &\qquad\times\left(\Pi_z\left(\exp\left\{\int_0^{t-s}(m-1)\xi(B_r)\mathrm{d}r\right\}1_{\{B_r\leq\varphi_t(r+s),\ \forall 0\leq r\leq t-s, B_{t-s}\geq V_t+2\}}\right)\right)^2_{|z=B_s}\Big)\mathrm{d}s \\ &\leq \mathrm{E}_y^\xi\left(\#\left\{\nu\in N(t) : X_t^\nu \geq V_t+2\right\}\right) \\ &\qquad + t^{\Gamma_0\lambda^* v^*}(m_2-m)\text{es}\int_0^t\Pi_y\Big(\exp\left\{\int_0^s(m-1)\xi(B_r)\mathrm{d}r\right\}1_{\{B_r<\varphi_t(r),\ \forall 0\leq r\leq s\}} \\ &\qquad\times\Pi_z\left(\exp\left\{\int_0^{t-s}(m-1)\xi(B_r)\mathrm{d}r\right\}1_{\{B_r\leq\varphi_t(r+s),\ \forall 0\leq r\leq t-s, B_{t-s}\geq V_t+2\}}\right)_{|z=B_s}\Big)\mathrm{d}s\end{aligned}$$

$$\leq e^{K_1} + t^{\Gamma_0\lambda^* v^*}(m_2 - m)\text{es}\int_0^t \Pi_y\left(\exp\left\{\int_0^t (m-1)\xi(B_r)\mathrm{d}r\right\} 1_{\{B_t \geq V_t+2\}}\right)\mathrm{d}s$$
$$\leq e^{K_1} + t^{\Gamma_0\lambda^* v^*+1} e^{K_1}(m_2 - m)\text{es}.$$

Taking $\ell_2 > \Gamma_0\lambda^* v^* + 1$, we arrive at the desired conclusion. □

Now we estimate $\mathrm{E}_y^\xi(\mathcal{L}_t)$ from below. Define

$$\begin{aligned} G_t &:= \left\{H_k \geq \frac{1}{\lambda^* v^*}\ln\psi(k+1, -\lambda^*) - \Gamma_0 \ln t,\ \forall k = 1, 2, ..., [V_t]\right\}; \\ A(r,t) &:= \left\{B_t \geq V_t + 2, \min_{s\in[t-1,t]} B_s \geq V_t - 3,\ B_s < \varphi_t(s),\ \forall s \in [r,t]\right\}; \\ \Gamma_1 &:= \frac{K_1}{\lambda^* v^*} + 2. \end{aligned} \tag{3.8}$$

For $y \in [-1, 1]$, by Proposition 1.2,

$$\begin{aligned} \mathrm{E}_y^\xi(\mathcal{L}_t) &= \Pi_y\left(\exp\left\{\int_0^t (m-1)\xi(B_s)\mathrm{d}s\right\}; A\left(H_{[V_t]}, t\right), G_t\right) \\ &\geq \Pi_y\left(\exp\left\{\int_0^t (m-1)\xi(B_s)\mathrm{d}s\right\}; A\left(H_{[V_t]}, t\right), H_{[V_t]} \in [t-\Gamma_1, t-1], G_t\right) \\ &\geq \Pi_y\left(\exp\left\{\int_0^{H_{[V_t]}} (m-1)\xi(B_s)\mathrm{d}s\right\}; A\left(H_{[V_t]}, t\right), H_{[V_t]} \in [t-\Gamma_1, t-1], G_t\right). \end{aligned} \tag{3.9}$$

By the definition of $\varphi_t(s)$, we know that

$$\frac{1}{\lambda^* v^*}\ln\psi(\varphi_t(s), -\lambda^*) - \Gamma_0 \ln t \leq s = \frac{1}{\lambda^* v^*}\ln\psi(V_s, -\lambda^*) < \frac{1}{\lambda^* v^*}\ln\psi(\varphi_t(s)+1, -\lambda^*) - \Gamma_0 \ln t.$$

It follows from Lemma 2.3(2) that for $s \in [0, t]$,

$$\frac{\Gamma_0\lambda^* v^*}{K_1}\ln t - 1 \leq \varphi_t(s) - V_s \leq \frac{\Gamma_0\lambda^* v^*}{K_2}\ln t. \tag{3.10}$$

By (2.20), when $r \in [t-\Gamma_1, t-1]$ and $s \in [r, t]$, $0 \leq V_t - V_s \leq \frac{\lambda^* v^*}{K_2}(t-s) \leq \frac{\lambda^* v^*}{K_2}\Gamma_1$. Therefore, applying (3.10) for $s \in [r, t]$, we have

$$\begin{aligned} A(r,t) &\supset \left\{B_t \geq V_t + 2, \min_{s\in[t-1,t]} B_s \geq V_t - 3,\ B_s < V_s + \frac{\Gamma_0\lambda^* v^*}{K_1}\ln t - 1,\ \forall s \in [r,t]\right\} \\ &\supset \left\{B_t \geq V_t + 2, \min_{s\in[t-1,t]} B_s \geq V_t - 3,\ B_s < V_t + \frac{\Gamma_0\lambda^* v^*}{K_1}\ln t - 1 - \frac{\lambda^* v^*}{K_2}\Gamma_1,\ \forall s \in [r,t]\right\} \\ &\supset \left\{B_t \geq [V_t]+3, \min_{s\in[t-1,t]} B_s \geq [V_t] - 2,\ B_s < [V_t] + \frac{\Gamma_0\lambda^* v^*}{K_1}\ln t - 1 - \frac{\lambda^* v^*}{K_2}\Gamma_1,\ \forall s \in [r,t]\right\}. \end{aligned}$$

Hence, when $r = H_{[V_t]} \in [t-\Gamma_1, t-1]$ and t is large enough so that $\frac{\Gamma_0\lambda^* v^*}{K_1}\ln t - 1 - \frac{\lambda^* v^*}{K_2}\Gamma_1 > 10$,

$$A(H_{[V_t]}, t) \supset \left\{B_t \geq B_r + 3, \min_{s\in[t-1,t]} B_s \geq B_r - 2,\ B_s < B_r + 10,\ \forall s \in [r,t]\right\}\Big|_{r=H_{[V_t]}}.$$

Using (3.9) and the strong Markov property of Brownian motion B, we have for t large,

$$\mathrm{E}_y^\xi[\mathcal{L}_t] \geq \Pi_y\left(\exp\left\{\int_0^{H_{[V_t]}} (m-1)\xi(B_s)\mathrm{d}s\right\} F_t; H_{[V_t]} \in [t-\Gamma_1, t-1], G_t\right), \tag{3.11}$$

where F_t is defined by

$$F_t := \Pi_0 \left(B_{t-r} \geq 3, \min_{s\in[t-r-1,t-r]} B_s \geq -2,\ B_s < 10,\ \forall s \in [0, t-r] \right)_{|r=H_{[V_t]}}.$$

For $r = H_{[V_t]} \in [t-\Gamma_1, t-1]$,

$$F_t \geq \inf_{r\in[1,\Gamma_1]} \Pi_0 \left(B_r \geq 3, \min_{s\in[r-1,r]} B_s \geq -2, \sup_{s\in[0,r]} B_s \leq 10 \right) =: \Gamma_2 > 0.$$

Plugging the lower bound of F_t into (3.11), using (2.2) with $\lambda = -\lambda^*$, for large t, we have that

$$\begin{aligned}
&\mathrm{E}_y^\xi(\mathcal{L}_t) \geq \Gamma_2 \Pi_y \left(\exp\left\{ \int_0^{H_{[V_t]}} (m-1)\xi(B_s)\mathrm{d}s \right\}; H_{[V_t]} \in [t-\Gamma_1, t-1], G_t \right) \\
&= \Gamma_2 \widetilde{\Pi}_y^{\xi,-\lambda^*} \left(e^{\lambda^* v^* H_{[V_t]}} \frac{\psi(y,-\lambda^*)}{\psi([V_t],-\lambda^*)}; H_{[V_t]} \in [t-\Gamma_1, t-1], G_t \right) \\
&\geq \Gamma_2 \widetilde{\Pi}_y^{\xi,-\lambda^*} \left(e^{\lambda^* v^* (t-\Gamma_1)} \frac{\psi(-1,-\lambda^*)}{\psi(V_t,-\lambda^*)}; H_{[V_t]} \in [t-\Gamma_1, t-1], G_t \right) \\
&\geq \Gamma_2 e^{-\lambda^* v^* \Gamma_1} e^{-K_1} \widetilde{\Pi}_y^{\xi,-\lambda^*} \left(H_{[V_t]} \in [t-\Gamma_1, t-1], G_t \right) \\
&\geq \Gamma_3 \widetilde{\Pi}_y^{\xi,-\lambda^*} \left(H_{[V_t]} \in [t-\Gamma_1, t-1], H_k \geq \frac{1}{\lambda^* v^*} \ln \psi(k,-\lambda^*) - (\Gamma_0 - 1)\ln t,\ 1 \leq k \leq [V_t] \right).
\end{aligned} \tag{3.12}$$

Here in the penultimate inequality, we used $\ln\psi(V_t,-\lambda^*) = \gamma(\lambda^*)t$ and (2.19) with $z=0, y=-1$, and in the last, we used $\Gamma_3 := \Gamma_2 e^{-\lambda^* v^* \Gamma_1} e^{-K_1}$ and t is large enough so that $\lambda^* v^* \ln t > K_1$.

To continue the estimate of $\mathrm{E}_y^\xi(\mathcal{L}_t)$, we will consider two independent copies of $(\Xi_t, \widetilde{\Pi}_y^{\xi,-\lambda^*})$. For $y \in [-1,1]$, we enlarge the probability space and the corresponding measure $\widetilde{\Pi}_y^{\xi,-\lambda^*}$: let $\Xi_t^j, j=1,2$ be two independent copies of $(\Xi_t, \widetilde{\Pi}_y^{\xi,-\lambda^*})$ and and

$$H_x^j := \inf\left\{ t \geq 0 :\ \Xi_t^j = x \right\}, \quad j = 1,2, x \in \mathbb{R}.$$

For $j = 1,2$, define

$$\beta_0^j := 0, \quad \beta_k^j := H_k^j - \frac{1}{\lambda^* v^*} \ln \psi(k,-\lambda^*), \quad k \geq 1. \tag{3.13}$$

Note that $\lambda^* v^* t - K_1 \leq \ln\psi([V_t],-\lambda^*) \leq \lambda^* v^* t$, we have

$$\left[\frac{1}{\lambda^* v^*} \ln\psi([V_t],-\lambda^*) + \frac{K_1}{\lambda^* v^*} - \Gamma_1, \frac{1}{\lambda^* v^*} \ln\psi([V_t],-\lambda^*) - 1 \right] \subset [t-\Gamma_1, t-1].$$

Together with the definition of Γ_1 in (3.8), we continue the estimate of (3.12) and get

$$\mathrm{E}_y^\xi(\mathcal{L}_t) \geq \Gamma_3 \widetilde{\Pi}_y^{\xi,-\lambda^*} \left(\beta_{[V_t]}^j \in [-2,-1], \beta_k^j \geq -(\Gamma_0 - 1)\ln t,\ 1 \leq k \leq [V_t] \right), j = 1,2.$$

We first claim that, under $\widetilde{\Pi}_y^{\xi,-\lambda^*}$, $\{\beta_k^1, k \geq 0\}$ has independent increments. The proof of this claim is postponed to the Appendix. Let $\pi := \pi_t$ be a uniform random variable on $\{2, ..., [V_t]-1\}$ which is independent of Ξ_t, Ξ_t^1 and Ξ_t^2. Define

$$\beta_k^{(t)} := \begin{cases} \beta_k^1, & 1 \leq k \leq \pi, \\ \beta_\pi^1 + \left(\beta_k^2 - \beta_\pi^2\right), & \pi < k \leq [V_t]. \end{cases}$$

We claim that

$$\left(\beta_k^{(t)}, k=1,...,[V_t];\ \widetilde{\Pi}_y^{\xi,-\lambda^*}\right) \overset{\mathrm{d}}{=} \left(\beta_k^1, k=1,...,[V_t];\ \widetilde{\Pi}_y^{\xi,-\lambda^*}\right). \tag{3.14}$$

In fact, for real numbers $\alpha_k, k=1,...,[V_t]$, we have

$$\begin{aligned}
&\widetilde{\Pi}_y^{\xi,-\lambda^*}\left(\exp\left\{\mathrm{i}\sum_{k=1}^{[V_t]}\alpha_k\beta_k^{(t)}\right\}\middle|\beta_k^1,\beta_k^2\right) \\
&= \frac{1}{[V_t]-2}\sum_{m=2}^{[V_t]-1}\exp\left\{\mathrm{i}\sum_{k=1}^{m}\alpha_k\beta_k^1 + \mathrm{i}\sum_{k=m+1}^{[V_t]}\alpha_k\left(\beta_m^1+\beta_k^2-\beta_m^2\right)\right\}.
\end{aligned}$$

Thus, it suffices to show for any $2 \le m \le [V_t]-1$,

$$\widetilde{\Pi}_y^{\xi,-\lambda^*}\left(\exp\left\{\mathrm{i}\sum_{k=1}^{m}\alpha_k\beta_k^1 + \mathrm{i}\sum_{k=m+1}^{[V_t]}\alpha_k\left(\beta_m^1+\beta_k^2-\beta_m^2\right)\right\}\right) = \widetilde{\Pi}_y^{\xi,-\lambda^*}\left(\exp\left\{\mathrm{i}\sum_{k=1}^{[V_t]}\alpha_k\beta_k^1\right\}\right). \tag{3.15}$$

Since under $\widetilde{\Pi}_y^{\xi,-\lambda^*}$, $\beta_k^j, j=1,2$, are sums of independent random variables, we know that, for $k>m, \beta_k^1-\beta_m^1$ is independent of $\beta_r^1, r\le m$. Use this observation, the left-hand side of (3.15) is

$$\begin{aligned}
&\widetilde{\Pi}_y^{\xi,-\lambda^*}\left(\exp\left\{\mathrm{i}\sum_{k=1}^{m}\alpha_k\beta_k^1 + \mathrm{i}\sum_{k=m+1}^{[V_t]}\alpha_k\left(\beta_m^1+\beta_k^2-\beta_m^2\right)\right\}\right) \\
&= \widetilde{\Pi}_y^{\xi,-\lambda^*}\left(\exp\left\{\mathrm{i}\sum_{k=1}^{m}\alpha_k\beta_k^1 + \mathrm{i}\sum_{k=m+1}^{[V_t]}\alpha_k\beta_m^1\right\}\right)\cdot\widetilde{\Pi}_y^{\xi,-\lambda^*}\left(\exp\left\{\mathrm{i}\sum_{k=m+1}^{[V_t]}\alpha_k\left(\beta_k^2-\beta_m^2\right)\right\}\right) \\
&= \widetilde{\Pi}_y^{\xi,-\lambda^*}\left(\exp\left\{\mathrm{i}\sum_{k=1}^{m}\alpha_k\beta_k^1 + \mathrm{i}\sum_{k=m+1}^{[V_t]}\alpha_k\beta_m^1\right\}\right)\cdot\widetilde{\Pi}_y^{\xi,-\lambda^*}\left(\exp\left\{\mathrm{i}\sum_{k=m+1}^{[V_t]}\alpha_k\left(\beta_k^1-\beta_m^1\right)\right\}\right) \\
&= \widetilde{\Pi}_y^{\xi,-\lambda^*}\left(\exp\left\{\mathrm{i}\sum_{k=1}^{m}\alpha_k\beta_k^1 + \mathrm{i}\sum_{k=m+1}^{[V_t]}\alpha_k\left(\beta_m^1+\beta_k^1-\beta_m^1\right)\right\}\right).
\end{aligned}$$

Hence we obtain (3.15), and thus the claim (3.14) holds.

Therefore, we have

$$\mathrm{E}_y^\xi(\mathcal{L}_t) \ge \Gamma_3\widetilde{\Pi}_y^{\xi,-\lambda^*}\left(\beta_{[V_t]}^{(t)}\in[-2,-1], \beta_k^{(t)} \ge -(\Gamma_0-1)\ln t,\ 1\le k\le[V_t]\right). \tag{3.16}$$

Now we are going to give a lower bound of the right hand side of (3.16). For a constant $C(\ell')$ to be specified in Lemma 3.2(i) below, define

$$\begin{aligned}
I_1 &:= \left\{\beta_k^1-\beta_2^1\ge 0,\ \forall 2\le k\le[V_t],\ \beta_{[V_t]}^1-\beta_2^1\ge[V_t]^{1/4}\right\}; \\
I_2 &:= \left\{\beta_k^2-\beta_{[V_t]}^2\ge 0,\ \forall 2\le k\le[V_t],\ \beta_2^2-\beta_{[V_t]}^2\ge[V_t]^{1/4}\right\}; \\
I_3 &:= \left\{\max_{3\le k\le[V_t]}\left|\beta_k^1-\beta_{k-1}^1\right|\le C(\ell')\ln[V_t]\right\}; \\
I_4 &:= \left\{\max_{3\le k\le[V_t]}\left|\beta_k^2-\beta_{k-1}^2\right|\le C(\ell')\ln[V_t]\right\}; \\
I_5 &:= \left\{\beta_2^1\in[-2+\beta_J^2-\beta_{[V_t]}^2-\beta_J^1+\beta_2^1, -1+\beta_J^2-\beta_{[V_t]}^2-\beta_J^1+\beta_2^1]\right\}\bigcap\left\{\pi=J\right\}.
\end{aligned} \tag{3.17}$$

Here $J := \sup\{j \le [V_t] : -2 + \beta_j^2 - \beta_{[V_t]}^2 - \beta_j^1 + \beta_2^1 \ge 0\}$. To give a lower bound of (3.16), we will prove that for sufficiently large t,

$$\bigcap_{n=1}^{5} I_n \subset \left\{\beta_{[V_t]}^{(t)} \in [-2,-1], \beta_k^{(t)} \ge -(\Gamma_0 - 1)\ln t,\ 1 \le k \le [V_t]\right\}. \tag{3.18}$$

By the definition of $\beta_k^{(t)}$, we know that, when $\bigcap_{n=1}^5 I_n$ occurs, we have $\beta_k^{(t)} = \beta_k^1$ if $1 \le k \le \pi = J$ and $\beta_k^{(t)} = \beta_J^1 + \beta_k^2 - \beta_J^2$ if $J < k \le [V_t]$. Note that on $I_1 \cap I_2$ we have

$$\beta_{[V_t]}^1 - \beta_2^1 \ge [V_t]^{1/4}, \qquad \beta_2^2 - \beta_{[V_t]}^2 \ge [V_t]^{1/4}.$$

This implies that for t large enough so that $[V_t]^{1/4} > 2$,

$$-2 + \beta_2^2 - \beta_{[V_t]}^2 - \beta_2^1 + \beta_2^1 > 0 \text{ and } -2 + \beta_{[V_t]}^2 - \beta_{[V_t]}^2 - \beta_{[V_t]}^1 + \beta_2^1 < 0,$$

which implies that $2 \le J < [V_t]$. When $k \le J$, then on $I_1 \cap I_3$ we have for $k \ge 2$,

$$\beta_k^{(t)} = \beta_k^1 \ge \beta_2^1 = H_2^1 - \frac{1}{\lambda^* v^*}\ln\psi(2,-\lambda^*) \ge -\frac{1}{\lambda^* v^*}\ln\psi(2,-\lambda^*) \ge -\frac{2K_1}{\lambda^* v^*},$$

and for $k = 1$, $\beta_k^{(t)} = \beta_1^1 = H_1^1 - (\lambda^* v^*)^{-1}\ln\psi(1,-\lambda^*) \ge -K_1/(\lambda^* v^*)$. Choose $\Gamma_0 > 2C(\ell')+1$. For large t, we have $-2K_1/(\lambda^* v^*) > -(\Gamma_0 - 1)\ln t$ and hence $\beta_k^{(t)} > -(\Gamma_0 - 1)\ln t$. Similarly, when $J < k \le [V_t]$, since $-2 + \beta_{J+1}^2 - \beta_{[V_t]}^2 - \beta_{J+1}^1 + \beta_2^1 < 0$ and $J + 1 \ge 3$, on $I_2 \cap I_3 \cap I_4$,

$$\begin{aligned}\beta_k^{(t)} &= \beta_J^1 + \beta_k^2 - \beta_J^2 \ge \beta_J^1 + \beta_{[V_t]}^2 - \beta_J^2 \ge -2C(\ell')\ln[V_t] + \beta_{J+1}^1 + \beta_{[V_t]}^2 - \beta_{J+1}^2 \\ &> -2C(\ell')\ln[V_t] + \beta_2^1 - 2 \ge -2C(\ell')\ln[V_t] - \frac{2K_1}{\lambda^* v^*} - 2.\end{aligned}$$

Since $\Gamma_0 > 2C(\ell') + 1$, we get that for large t, $\beta_k^{(t)} > -(\Gamma_0 - 1)\ln t$. Note that $\beta_{[V_t]}^{(t)} = \beta_J^1 + \beta_{[V_t]}^2 - \beta_J^2$ and by the definition of I_5, $\beta_{[V_t]}^{(t)} \in [-2,-1]$ and hence we get (3.18).

By (3.16) and (3.18), we have

$$\mathrm{E}_y^\xi(\mathcal{L}_t) \ge \Gamma_3 \widetilde{\Pi}_y^{\xi,-\lambda^*}\left(\cap_{j=1}^5 I_j\right) = \Gamma_3 \widetilde{\Pi}_y^{\xi,-\lambda^*}\left(\cap_{j=1}^4 I_j\right)\cdot \widetilde{\Pi}_y^{\xi,-\lambda^*}\left(I_5 \middle| \cap_{j=1}^4 I_j\right). \tag{3.19}$$

Since I_1 and I_2 are independent, we have

$$\begin{aligned}\widetilde{\Pi}_y^{\xi,-\lambda^*}\left(\cap_{j=1}^4 I_j\right) &\ge \widetilde{\Pi}_y^{\xi,-\lambda^*}(I_1 \cap I_2) + \widetilde{\Pi}_y^{\xi,-\lambda^*}(I_3 \cap I_4) - 1 \\ &\ge \widetilde{\Pi}_y^{\xi,-\lambda^*}(I_1)\cdot\widetilde{\Pi}_y^{\xi,-\lambda^*}(I_2) + \widetilde{\Pi}_y^{\xi,-\lambda^*}(I_3 \cap I_4) - 1.\end{aligned} \tag{3.20}$$

By our choice of J, conditioned on $\cap_{j=1}^4 I_j$, we know that

$$x_1 := -2 + \beta_J^2 - \beta_{[V_t]}^2 - \beta_J^1 + \beta_2^1 \in [0, 2C(\ell')\ln[V_t]], \tag{3.21}$$

and x_1 is independent of β_2^1. Also note that β_2^1 is independent of $\cap_{j=1}^4 I_j$ by Lemma 4.1. Thus,

$$\widetilde{\Pi}_y^{\xi,-\lambda^*}\left(I_5 \middle| \cap_{j=1}^4 I_j\right) \ge \frac{1}{[V_t] - 2}\inf_{x_1 \in [0, 2C(\ell')\ln[V_t]]}\widetilde{\Pi}_y^{\xi,-\lambda^*}\left(\beta_2^1 \in [x_1, x_1 + 1]\right). \tag{3.22}$$

Thus, to continue the estimate in (3.19), we need to estimate $\widetilde{\Pi}_y^{\xi,-\lambda^*}(I_3 \cap I_4)$,

$\widetilde{\Pi}_y^{\xi,-\lambda^*}\left(\beta_2^1 \in [x, x+1]\right)$ and $\widetilde{\Pi}_y^{\xi,-\lambda^*}(I_j)$ for $j = 1, 2$. Now we first estimate $\widetilde{\Pi}_y^{\xi,-\lambda^*}(I_3 \cap I_4)$ and $\widetilde{\Pi}_y^{\xi,-\lambda^*}\left(\beta_2^1 \in [x, x+1]\right)$.

Lemma 3.2. *(i) For any $\ell'>0$, there exists a non-random constant $C(\ell')>0$ so that for large t,*

$$\inf_{y\in[-1,1]}\widetilde{\Pi}_y^{\xi,-\lambda^*}\left(I_3\cap I_4\right)\geq 1-[V_t]^{-\ell'}. \tag{3.23}$$

(ii) There exists a constant $C_3>0$ such that for all $x>0$,

$$\inf_{y\in[-1,1]}\inf_{x_1\in[0,x]}\widetilde{\Pi}_y^{\xi,-\lambda^*}\left(\beta_2^1\in[x_1,x_1+1]\right)\geq C_3e^{-x/C_3}.$$

Proof. (i) By the definition of β_k^j and Lemma 2.3(2), for $k\geq 3$,

$$\left|\beta_k^j-\beta_{k-1}^j\right|\leq H_k^j-H_{k-1}^j+\frac{K_1}{\lambda^*v^*}.$$

For any $0\leq\eta<\gamma(-\lambda^*)-(m-1)\text{es}=\gamma(\lambda^*)-(m-1)\text{es}$, by the strong Markov property of Ξ^j,

$$\begin{aligned}
&\widetilde{\Pi}_y^{\xi,-\lambda^*}\left(\exp\left\{\eta(H_k^j-H_{k-1}^j)\right\}\right)=\widetilde{\Pi}_{k-1}^{\xi,-\lambda^*}\left(\exp\left\{\eta H_k^j\right\}\right)\\
&=\Pi_{k-1}\left(\exp\left\{\int_0^{H_k}((m-1)\xi(B_s)-\gamma(-\lambda^*))\,\mathrm{d}s+\eta H_k\right\}\frac{\psi(k,-\lambda^*)}{\psi(k-1,-\lambda^*)}\right)\\
&\leq e^{K_1}\Pi_{k-1}\left(\exp\left\{((m-1)\text{es}-\gamma(-\lambda^*)+\eta)\,H_k\right\}\right)\\
&=\exp\left\{-\sqrt{2\left(\gamma(-\lambda^*)-(m-1)\text{es}-\eta\right)}+K_1\right\}=:c_1<+\infty.
\end{aligned}\tag{3.24}$$

Hence, for fixed $\eta\in(0,\gamma(\lambda^*)-(m-1)\text{es})$, we have for all $y\in[-1,1]$,

$$\begin{aligned}
&\widetilde{\Pi}_y^{\xi,-\lambda^*}\left((I_3\cap I_4)^c\right)=\widetilde{\Pi}_y^{\xi,-\lambda^*}\left(\max_{3\leq k\leq[V_t]}\max_{j=1,2}\left|\beta_k^j-\beta_{k-1}^j\right|>C(\ell')\ln[V_t]\right)\\
&\leq\sum_{k=3}^{[V_t]}\sum_{j=1}^{2}\widetilde{\Pi}_y^{\xi,-\lambda^*}\left(\left|\beta_k^j-\beta_{k-1}^j\right|>C(\ell')\ln[V_t]\right)\\
&\leq\sum_{k=3}^{[V_t]}\sum_{j=1}^{2}e^{-\eta C(\ell')\ln[V_t]}\widetilde{\Pi}_y^{\xi,-\lambda^*}\left(\exp\left\{\eta\left|\beta_k^j-\beta_{k-1}^j\right|\right\}\right)\leq 2c_1[V_t]^{1-\eta C(\ell')}\cdot e^{\eta K_1/(\lambda^*v^*)}.
\end{aligned}$$

Taking $C(\ell')$ large so that $\eta C(\ell')-1>\ell'$, we get that (3.23) holds for sufficiently large t.

(ii) For simplicity, we set $\gamma_0:=\gamma(-\lambda^*)-(m-1)\text{ei}$. By the definition of β_2^1 and noting that $0\leq x_0:=\frac{1}{\lambda^*v^*}\ln\psi(2,-\lambda^*)\leq 2K_1/(\lambda^*v^*)$, for $x_1\in[0,x]$, we have

$$\begin{aligned}
&\widetilde{\Pi}_y^{\xi,-\lambda^*}\left(\beta_2^1\in[x_1,x_1+1]\right)=\widetilde{\Pi}_y^{\xi,-\lambda^*}\left(H_2\in[x_1+x_0,x_1+x_0+1]\right)\\
&=\Pi_y\left(\exp\left\{\int_0^{H_2}((m-1)\xi(B_s)-\gamma(-\lambda^*))\,\mathrm{d}s\right\};H_2\in[x_1+x_0,x_1+x_0+1]\right)\\
&\geq\Pi_y\left(e^{-\gamma_0H_2};H_2\in[x_1+x_0,x_1+x_0+1]\right)\\
&\geq e^{-\gamma_0(x_1+x_0+1)}\Pi_0\left(H_{2-y}\in[x_1+x_0,x_1+x_0+1]\right)\\
&\geq e^{-\gamma_0(x+2K_1/(\lambda^*v^*)+1)}\Pi_0\left(H_{2-y}\in[x_1+x_0+1/2,x_1+x_0+1]\right).
\end{aligned}$$

Since H_{2-y} has density $p_{H_{2-y}}(s):=\frac{(2-y)}{\sqrt{2\pi s^3}}e^{-(2-y)^2/(2s)}$, we have

$$\begin{aligned}
&\Pi_0\left(H_{2-y}\in[x_1+x_0+1/2,x_1+x_0+1]\right)=\int_{x_1+x_0+1/2}^{x_1+x_0+1}\frac{(2-y)}{\sqrt{2\pi s^3}}e^{-(2-y)^2/(2s)}\mathrm{d}s\\
&\geq\frac{1}{2}\frac{1}{\sqrt{2\pi\left(x_1+1+2K_1/(\lambda^*v^*)\right)^3}}e^{-9}\geq\frac{1}{2}\frac{1}{\sqrt{2\pi\left(x+1+2K_1/(\lambda^*v^*)\right)^3}}e^{-9}.
\end{aligned}$$

Since $z < e^z$ for $z > 0$, we have $z^{-3/2} \geq e^{-3z/2}$ for $z > 0$, and thus there exist $c_2, c_3 > 0$ such that

$$\inf_{y\in[-1,1]} \inf_{x_1\in[0,x]} \widetilde{\Pi}_y^{\xi,-\lambda^*} \left(\beta_1^1 \in [x_1, x_1+1]\right) \geq c_2 e^{-c_3 x}, \quad x > 0.$$

Taking $C_3 = \min\{c_2, 1/c_3\}$, we arrive at the desired conclusion. □

The estimates of $\widetilde{\Pi}_y^{\xi,-\lambda^*}(I_1)$ and $\widetilde{\Pi}_y^{\xi,-\lambda^*}(I_2)$ are given below.

Lemma 3.3. *There exists a non-random $\ell'' > 0$ such that for t large,*

$$\inf_{y\in[-1,1]} \widetilde{\Pi}_y^{\xi,-\lambda^*}(I_1) \geq [V_t]^{-\ell''}; \quad \inf_{y\in[-1,1]} \widetilde{\Pi}_y^{\xi,-\lambda^*}(I_2) \geq [V_t]^{-\ell''}.$$

We postpone the proof of Lemma 3.3 to Subsection 3.4.

Now we are ready to prove the following estimate on the first moment of $\mathcal{L}_t$:

Lemma 3.4. *There exists a non-random constant $\ell_1 > 0$ such that for t large,*

$$\inf_{y\in[-1,1]} \mathrm{E}_y^\xi(\mathcal{L}_t) \geq t^{-\ell_1}.$$

Proof. Recall the definition of x_1 in (3.21). By (3.22) and Lemma 3.2(ii),

$$\widetilde{\Pi}_y^{\xi,-\lambda^*}\left(I_5 \middle| \cap_{j=1}^4 I_j\right) \geq \frac{1}{[V_2]-2} C_3 e^{-2C(\ell')\ln[V_t]/C_3} \geq C_3 [V_t]^{-\frac{2C(\ell')}{C_3}-1}.$$

By (3.20), Lemma 3.2(i) and Lemma 3.3, we have

$$\inf_{y\in[-1,1]} \widetilde{\Pi}_y^{\xi,-\lambda^*}\left(\cap_{j=1}^4 I_j\right) \geq [V_t]^{-2\ell''} - [V_t]^{-\ell'}. \tag{3.25}$$

Note that $\ell' > 0$ in Lemma 3.2 is arbitrary. Letting $\ell' > 2\ell''$ and using (3.19), (3.22) and (3.25), we finally get that

$$\inf_{y\in[-1,1]} \mathrm{E}_y^\xi(\mathcal{L}_t) \geq \Gamma_3 C_3 [V_t]^{-2C(\ell')/C_3-1-2\ell''} \cdot \left(1 - [V_t]^{-(\ell'-2\ell'')}\right).$$

Combining this with the fact that $V_t/t \to v^*$, and letting $\ell_1 > 2\ell'' + 2C(\ell')/C_3 + 1$, we get the desired result. □

Lemma 3.5. *There exists a non-random constant $C_4 > 1$ such that for sufficiently large t,*

$$\sup_{x\in\mathbb{R}} \mathrm{P}_x^\xi\left(\#\{\nu \in N(t) : X_t^\nu \in [x-1, x+1]\} \leq C_4^t\right) \leq C_4^{-t}.$$

Proof. See [7, Lemma 6.8] for the case of continuous-time branching random walk in random environment or [10, Lemma 4.7] for the case of branching Brownian motion in random environment. □

Proof of Theorem 1.6. Let $\lambda = -\lambda^*$ and let Γ_4 be a constant such that $\Gamma_4 \ln C_4 > \ell_2 + 2\ell_1$, where C_4 is the constant in Lemma 3.5. For any $n \in \mathbb{N}$, define $r_n := [\Gamma_4 \ln n]$. For simplicity, we will omit the subscript n from r_n in this proof. By (2.20), $V_t - V_s \leq \frac{\lambda^* v^*}{K_2}(t-s)$ for $t > s$. Let $\Gamma = \lambda^* v^* \Gamma_4/K_2 + 1$, then $V_n - V_{n-r} \leq (\Gamma - 1)\ln n$ and

$$\begin{aligned}
&\mathrm{P}_0^\xi\left(\inf_{n-1\leq t\leq n} M_t - V_n < -\Gamma \ln n\right) \\
&\leq \mathrm{P}_0^\xi\left(\#\{\nu \in N(r) : |X_r^\nu| \leq 1\} \leq C_4^r\right) + \sup_{|y|\leq 1}\left\{\mathrm{P}_y^\xi\left(\inf_{n-1\leq t\leq n} M_{t-r} - V_n < -\Gamma \ln n\right)\right\}^{C_4^r} \\
&\leq C_4 n^{-\Gamma_4 \ln C_4} + \sup_{|y|\leq 1}\left\{\mathrm{P}_y^\xi\left(\inf_{n-1\leq t\leq n} M_{t-r} - V_{n-r} < -\ln n\right)\right\}^{n^{\Gamma_4 \ln C_4}/C_4}.
\end{aligned} \tag{3.26}$$

Recall the definition of $\mathcal{L}_t$ given by (3.6). For large n such that $\ln n > 3$, we have

$$\mathrm{P}_y^\xi\left(\inf_{n-1\le t\le n} M_{t-r} - V_{n-r} \ge -\ln n\right) \ge \mathrm{P}_y^\xi\left(\inf_{n-1\le t\le n} M_{t-r} - V_{n-r} \ge -3\right) \\ \ge \mathrm{P}_y^\xi(\mathcal{L}_{n-r} > 0). \tag{3.27}$$

On the other hand, by Lemmas 3.1 and 3.4, there exists n_0 such that for all $n \ge n_0$ and $y \in [-1, 1]$,

$$\mathrm{P}_y^\xi\left(\mathcal{L}_{n-r} > 0\right) \ge \frac{\left(\mathrm{E}_y^\xi\left(\mathcal{L}_{n-r}\right)\right)^2}{\mathrm{E}_y^\xi\left((\mathcal{L}_{n-r})^2\right)} \ge (n-r)^{-\ell_2-2\ell_1} \ge n^{-\ell_2-2\ell_1}. \tag{3.28}$$

Combining (3.26), (3.27) and (3.28), we get

$$\mathrm{P}_0^\xi\left(\inf_{n-1\le t\le n} M_t - V_n < -\Gamma\ln n\right) \le C_4 n^{-\Gamma_4\ln C_4} + \left(1 - n^{-\ell_2-2\ell_1}\right)^{n^{\Gamma_4\ln C_4}/C_4}.$$

Since $1 - x \le e^{-x}$ for $x > 0$, we have

$$\mathrm{P}_0^\xi\left(\inf_{n-1\le t\le n} M_t - V_n < -\Gamma\ln n\right) \le C_4 n^{-\Gamma_4\ln C_4} + \exp\left\{-n^{\Gamma_4\ln C_4-\ell_2-2\ell_1}/C_4\right\}.$$

Applying the Borel-Cantelli lemma, we get

$$\liminf_{n\to\infty}\frac{\inf_{n-1\le t\le n} M_t - V_n}{\ln n} \ge -\Gamma, \quad \mathrm{P}_0^\xi\text{-a.s.}.$$

Since $|V_n - V_t|$ is uniformly bounded for n and t satisfying $|n-t| \le 1$, we get that P_0^ξ-almost surely,

$$\liminf_{t\to\infty}\frac{M_t - V_t}{\ln t} \ge -\Gamma,$$

which is equivalent to

$$\limsup_{t\to\infty}\frac{V_t - M_t}{\ln t} \le \Gamma.$$

By Theorem 1.4, $\lim_{t\to\infty} W_t(-\lambda^*) = 0$. Since $W_t(-\lambda^*) \ge \psi(M_t, -\lambda^*)e^{-\gamma(\lambda^*)t}$, we have

$$\lim_{t\to\infty}\psi(M_t, -\lambda^*)e^{-\gamma(\lambda^*)t} = 0.$$

Therefore $\lim_{t\to\infty}(\ln\psi(M_t,-\lambda^*)) - \ln\psi(V_t,-\lambda^*)) = \lim_{t\to\infty}(-\gamma(\lambda^*)t + \ln\psi(M_t,-\lambda^*)) = -\infty$. By Remark 2.4, $M_t - V_t \to -\infty$, P_0^ξ-a.s. as $t \to \infty$, this completes the proof. □

3.3 Proof of Theorem 1.7

In this subsection, we also assume that **(H1)**, **(H2)** and **(H3)** hold.

Lemma 3.6. *Let $\kappa > 0$. Suppose that $W(x,\omega) : [0,\infty)\times\Omega \to \mathbb{R}$ is a continuous process on $(\Omega, \mathcal{F}, \mathbb{P})$ such that, $\mathbb{P}$-almost surely, $W(0,\omega) = 0$ and*

$$\lim_{x\to\infty}\frac{W(x,\omega)}{x} = 0.$$

Suppose further that the family of processes $\left\{n^{-1}W(nx,\omega)\right\}_{n=1}^\infty$ converges weakly as $n \to \infty$ to $\kappa B(x)$, where $B(x)$ is a standard Brownian motion on the interval $[0, M]$, for any M, in the topology of uniform convergence. Define

$$h_t(\omega) = \sup\left\{h \ge -\lambda^* v^* t \,\middle|\, W(h + \lambda^* v^* t) = h - 1\right\}.$$

Then as $t \to \infty$, the family of processes $\left\{\frac{1}{\kappa\sqrt{\lambda^* v^* n}} h_{nt}\right\}_{n=1}^\infty$ converges weakly as $n \to \infty$ to a standard Brownian motion on $[0, M]$, in the Skorohod space.

Proof. See [21, Lemma 3.1]. □

Lemma 3.7. *Let $\tilde{\sigma}^2_{-\lambda^*}$ be the constant defined in Theorem 1.7. Under $\mathbb{P}$, we have*

$$\frac{V_t - v^* t}{\sqrt{t}} \overset{\mathrm{d}}{\Rightarrow} \mathcal{N}\left(0, \tilde{\sigma}^2_{-\lambda^*}\right), \text{ as } t \to \infty.$$

If $\tilde{\sigma}^2_{-\lambda^*} > 0$, then the sequence of processes

$$[0,\infty) \ni t \mapsto \frac{V_{nt} - v^* nt}{\tilde{\sigma}_{-\lambda^*}\sqrt{n}}, \quad n \in \mathbb{N},$$

converges weakly as $n \to \infty$ to a standard Brownian motion on $[0,\infty)$, in the Skorohod topology.

Proof. Combining (2.1), Lemma 2.3(2) and the fact that $\psi(\cdot, -\lambda^*)$ is strictly increasing, we see that V_t is the unique solution to $\ln \psi(x, -\lambda^*) = \lambda^* v^* t$. Using Remark 2.7 and the fact that $\psi(\cdot, -\lambda^*)$ is strictly increasing, we get that, for any fixed $y \in \mathbb{R}$,

$$\begin{aligned}
\mathbb{P}\left(\frac{V_t - v^* t}{\sqrt{t}} \le y\right) &= \mathbb{P}\left(V_t \le v^* t + y\sqrt{t}\right) = \mathbb{P}\left(\lambda^* v^* t \le \ln \psi(v^* t + y\sqrt{t}, -\lambda^*)\right) \\
&= \mathbb{P}\left(-\ln \phi(v^* t + y\sqrt{t}, -\lambda^*) \le \lambda^* y \sqrt{t}\right) \\
&= \mathbb{P}\left(-\frac{\ln \phi(v^* t + y\sqrt{t}, -\lambda^*)}{\sqrt{v^* t + y\sqrt{t}}} \le \frac{\lambda^*}{\sqrt{v^*}} y \cdot \sqrt{\frac{t}{t + y\sqrt{t}/v^*}}\right) \\
&\overset{t\to\infty}{\longrightarrow} \mathbb{P}\left((\sigma'_{-\lambda^*})\chi \le \frac{\lambda^*}{\sqrt{v^*}} y\right) = \mathbb{P}((\tilde{\sigma}_{-\lambda^*})\chi \le y).
\end{aligned}$$

Here χ is a standard normal random variable.

If $\tilde{\sigma}^2_{-\lambda^*} > 0$, let $W(x,\omega) := -\ln \phi\left(\frac{x}{\lambda^*}, -\lambda^*, \omega\right)$ and $\kappa := \sigma'_{-\lambda^*}/\sqrt{\lambda^*} > 0$. It is easy to see that

$$h_t(\omega) = \lambda^* \left(V_{t+(\lambda^* v^*)^{-1}}(\omega) - v^* t\right).$$

Using Lemma 3.6, we have that the sequence of processes

$$[0,\infty) \ni t \mapsto \frac{h_{nt}}{\kappa\sqrt{\lambda^* v^* n}} = \frac{V_{nt+(\lambda^* v^*)^{-1}} - v^* nt}{\tilde{\sigma}_{-\lambda^*}\sqrt{n}}, \quad n \in \mathbb{N},$$

converges as $n \to \infty$ weakly to a standard Brownian motion on $[0, M]$, for any $M > 0$, in the Skorohod topology. According to [2, Lemma 3, p.173], this is equivalent to the weak convergence to a standard Brownian motion on $[0,\infty)$ in the Skorohod topology. By Remark 2.4, $\left|V_{t+(\lambda^* v^*)^{-1}} - V_t\right| \le 1/K_2$ all $t \ge 0$, hence the second conclusion of the lemma is valid. □

Proof of Theorem 1.7. This follows from Theorem 1.6 and Lemma 3.7. □

3.4 Proof of Lemma 3.3

In this subsection we give the proof of Lemma 3.3. We first prove several lemmas.

Lemma 3.8. *There exists a non-random constant $C_5 > 0$ such that for $j = 1, 2$,*

$$\inf_{y\in[-1,1],k\in\mathbb{N},k\ge 2} \widetilde{\Pi}^{\xi,-\lambda^*}_y \left(\beta^j_k - \beta^j_{k-1} > C_5\right) > C_5,$$

$$\inf_{y\in[-1,1],k\in\mathbb{N},k\ge 2} \widetilde{\Pi}^{\xi,-\lambda^*}_y \left(\beta^j_k - \beta^j_{k-1} < -C_5\right) > C_5.$$

Proof. Since β^1 and β^2 are identically distributed, we only prove the case $j=2$. Recall that $\beta^2_k = H^2_k - \frac{1}{\lambda^* v^*}\ln\psi(k,-\lambda^*)$ and $0 < K_2 \le \ln\psi(k,-\lambda^*) - \ln\psi(k-1,-\lambda^*) \le K_1$ for every $k\in\mathbb{Z}$ and $k\ge 2$. Note that for any Borel set $B\subset[0,+\infty)$ with $m(B)>0$,

$$\widetilde{\Pi}_y^{\xi,-\lambda^*}\left(H^2_k - H^2_{k-1}\in B\right) = \widetilde{\Pi}_{k-1}^{\xi,-\lambda^*}\left(H^2_k\in B\right) \ge \Pi_0\left(e^{-H_1(\gamma(\lambda^*)-(m-1)\mathrm{ei})}; H_1\in B\right).$$

For $0<\delta<K_2/(\lambda^* v^*)$, we have

$$\begin{aligned}\widetilde{\Pi}_y^{\xi,-\lambda^*}\left(\beta^2_k - \beta^2_{k-1} < -\delta\right) &\ge \widetilde{\Pi}_y^{\xi,-\lambda^*}\left(H^2_k - H^2_{k-1} < -\delta + \frac{K_2}{\lambda^* v^*}\right)\\ &\ge \Pi_0\left(e^{-H_1(\gamma(\lambda^*)-(m-1)\mathrm{ei})}; H_1 < -\delta + K_2/(\lambda^* v^*)\right).\end{aligned}$$

Similarly,

$$\widetilde{\Pi}_y^{\xi,-\lambda^*}\left(\beta^2_k - \beta^2_{k-1} > \delta\right) \ge \Pi_0\left(e^{-H_1(\gamma(\lambda^*)-(m-1)\mathrm{ei})}; H_1 > \delta + K_1/(\lambda^* v^*)\right).$$

When $\delta=0$, it holds that

$$\begin{aligned}&\Pi_0\left(e^{-H_1(\gamma(\lambda^*)-(m-1)\mathrm{ei})}; H_1 < K_2/(\lambda^* v^*)\right) > 0 = \delta,\\ &\Pi_0\left(e^{-H_1(\gamma(\lambda^*)-(m-1)\mathrm{ei})}; H_1 > K_1/(\lambda^* v^*)\right) > 0 = \delta.\end{aligned}$$

Since H_1 has a density, the maps $\delta\mapsto\Pi_0\left(e^{-H_1(\gamma(\lambda^*)-(m-1)\mathrm{ei})}; H_1 < -\delta + K_2/(\lambda^* v^*)\right)$ and $\delta\mapsto\Pi_0\left(e^{-H_1(\gamma(\lambda^*)-(m-1)\mathrm{ei})}; H_1 > \delta + K_1/(\lambda^* v^*)\right)$ are continuous in $\delta>0$. Thus there exists $C_5>0$ such that

$$\begin{aligned}&\Pi_0\left(e^{-H_1(\gamma(\lambda^*)-(m-1)\mathrm{ei})}; H_1 < -C_5 + K_2/(\lambda^* v^*)\right) > C_5,\\ &\Pi_0\left(e^{-H_1(\gamma(\lambda^*)-(m-1)\mathrm{ei})}; H_1 > C_5 + K_1/(\lambda^* v^*)\right) > C_5.\end{aligned}$$

This completes the proof. □

Lemma 3.9. *For any $\kappa_0>0$, there exist non-random constants $N(\kappa_0)>0$ and $C_6 = C_6(\kappa_0)\in(0,1)$ such that for all $r\ge 2, k\ge N(\kappa_0)$ and $j=1,2$,*

$$\inf_{y\in[-1,1]}\widetilde{\Pi}_y^{\xi,-\lambda^*}\left(\inf_{r<m\le r+k}\left(H^j_m - H^j_r - \widetilde{\Pi}_y^{\xi,-\lambda^*}\left(H^j_m - H^j_r\right)\right)\ge -\kappa_0\sqrt{k}\right)\ge C_6.$$

Proof. We only deal with $j=1$, the case $j=2$ is similar. Define

$$\Delta H_k := H^1_k - H^1_{k-1} - \widetilde{\Pi}_y^{\xi,-\lambda^*}\left(H^1_k - H^1_{k-1}\right) \tag{3.29}$$

and

$$f(\eta) := \ln\widetilde{\Pi}_y^{\xi,-\lambda^*}\left(\exp\left\{\eta(H^1_k - H^1_{k-1})\right\}\right).$$

Note that by (2.2), (2.11) and (2.7), we have

$$\begin{aligned}\widetilde{\Pi}_y^{\xi,-\lambda^*}\left(H^1_k\right) &= \Pi_y\left(H_k\exp\left\{-\gamma(-\lambda^*)H_k + \int_0^{H_k}(m-1)\xi(B_s)\mathrm{d}s\right\}\frac{\psi(k,-\lambda^*)}{\psi(y,-\lambda^*)}\right)\\ &= \frac{\psi(k,-\lambda^*,\omega)}{\psi(y,-\lambda^*,\omega)}\cdot\frac{1}{-\gamma'(-\lambda^*)}\frac{\partial}{\partial\lambda}\Pi_y\left(\exp\left\{-\gamma(\lambda)H_k + \int_0^{H_k}(m-1)\xi(B_s)\mathrm{d}s\right\}\right)\Big|_{\lambda=-\lambda^*}\\ &= \frac{\psi(k,-\lambda^*,\omega)}{\psi(y,-\lambda^*,\omega)}\cdot\frac{1}{-\gamma'(-\lambda^*)}\frac{\partial}{\partial\lambda}\left(\frac{\psi(y,\lambda,\omega)}{\psi(k,\lambda,\omega)}\right)\Big|_{\lambda=-\lambda^*}\end{aligned}$$

$$= \frac{\psi(k-y,-\lambda^*,\theta_y\omega)}{-\gamma'(-\lambda^*)}\frac{\partial}{\partial\lambda}\left(\frac{1}{\psi(k-y,\lambda,\theta_y\omega)}\right)\Big|_{\lambda=-\lambda^*} = \frac{1}{\gamma'(-\lambda^*)}\frac{\psi_\lambda(k-y,-\lambda^*,\theta_y\omega)}{\psi(k-y,-\lambda^*,\theta_y\omega)}$$
$$= \frac{1}{\gamma'(-\lambda^*)}\left[\frac{\psi_\lambda(k,-\lambda^*,\omega)}{\psi(k,-\lambda^*,\omega)} - \frac{\psi_\lambda(y,-\lambda^*,\omega)}{\psi(y,-\lambda^*,\omega)}\right].$$

Thus

$$\widetilde{\Pi}_y^{\xi,-\lambda^*}\left(H_k^1\right) - \widetilde{\Pi}_y^{\xi,-\lambda^*}\left(H_{k-1}^1\right) = \frac{1}{\gamma'(-\lambda^*)}\left(\frac{\psi_\lambda(k,-\lambda^*,\omega)}{\psi(k,-\lambda^*,\omega)} - \frac{\psi_\lambda(k-1,-\lambda^*,\omega)}{\psi(k-1,-\lambda^*,\omega)}\right).$$

Using Lemma 2.3(1), we obtain

$$\left|\widetilde{\Pi}_y^{\xi,-\lambda^*}\left(H_k^1\right) - \widetilde{\Pi}_y^{\xi,-\lambda^*}\left(H_{k-1}^1\right)\right| \le \frac{C_1(-\lambda^*)}{|\gamma'(-\lambda^*)|}, \tag{3.30}$$

here $C_1(-\lambda^*)$ is the constant in Lemma 2.3.

By (3.24), we have, uniformly for any $0 \le \eta \le (\gamma(\lambda^*) - (m-1)\text{es})/2$ and $k \ge 2$,

$$\begin{aligned} f'(\eta) &= \frac{\widetilde{\Pi}_y^{\xi,-\lambda^*}\left(\exp\left\{\eta(H_k^1 - H_{k-1}^1)\right\}(H_k^1 - H_{k-1}^1)\right)}{\widetilde{\Pi}_y^{\xi,-\lambda^*}\left(\exp\left\{\eta(H_k^1 - H_{k-1}^1)\right\}\right)} \\ &= \frac{\widetilde{\Pi}_{k-1}^{\xi,-\lambda^*}\left(\exp\left\{\eta H_k^1\right\}H_k^1\right)}{\widetilde{\Pi}_{k-1}^{\xi,-\lambda^*}\left(\exp\left\{\eta H_k^1\right\}\right)} \le \widetilde{\Pi}_{k-1}^{\xi,-\lambda^*}\left(\exp\left\{\eta H_k^1\right\}H_k^1\right) \\ &\le \Pi_0\left(\exp\left\{-H_1\left(\gamma(\lambda^*) - (m-1)\text{es} - \eta\right)\right\}H_1\right)\cdot e^{K_1} \\ &\le \Pi_0\left(\exp\left\{-H_1\left(\gamma(\lambda^*) - (m-1)\text{es}\right)/2\right\}H_1\right)\cdot e^{K_1} =: c_1 < +\infty. \end{aligned}$$

This implies that $f(\eta) \le \eta c_1$ for any $0 \le \eta \le (\gamma(\lambda^*) - (m-1)\text{es})/2$. Therefore, by (3.30),

$$\begin{aligned} \widetilde{\Pi}_y^{\xi,-\lambda^*}\left(e^{\eta|\Delta H_k|}\right) &\le \widetilde{\Pi}_y^{\xi,-\lambda^*}\left(\exp\left\{\eta(H_k^1 - H_{k-1}^1)\right\}\right)\cdot\exp\left\{\eta\left|\widetilde{\Pi}_y^{\xi,-\lambda^*}\left(H_k^1 - H_{k-1}^1\right)\right|\right\} \\ &\le \exp\left\{\eta c_1 + \eta\left|C_1(-\lambda^*)/\gamma'(-\lambda^*)\right|\right\}. \end{aligned}$$

Thus, ΔH_k are sub-exponential random variables for $k \ge 1$. By [25, Proposition 2.7.1], there exists c_2 depending only on $c_1 + C_1(-\lambda^*)/\left|\gamma'(-\lambda^*)\right|$ such that

$$\widetilde{\Pi}_y^{\xi,-\lambda^*}\left(e^{\eta\Delta H_k}\right) \le \exp\left\{(c_2\eta)^2\right\}, \quad |\eta| \le 1/c_2, k \ge 1. \tag{3.31}$$

By martingale theory and Doob's inequality, for $0 < \eta < 1/(2c_2)$,

$$\begin{aligned} &\widetilde{\Pi}_y^{\xi,-\lambda^*}\left(\inf_{r<m\le r+k}\sum_{\ell=r+1}^m \Delta H_\ell \ge -\kappa_0\sqrt{k}\right) = 1 - \widetilde{\Pi}_y^{\xi,-\lambda^*}\left(\sup_{r<m\le r+k}\sum_{\ell=r+1}^m(-\Delta H_\ell) > \kappa_0\sqrt{k}\right) \\ &= 1 - \widetilde{\Pi}_y^{\xi,-\lambda^*}\left(\sup_{r<m\le r+k}\exp\left\{\eta\sum_{\ell=r+1}^m(-\Delta H_\ell)\right\} > e^{\eta\kappa_0\sqrt{k}}\right) \\ &\ge 1 - e^{-2\eta\kappa_0\sqrt{k}}\cdot\widetilde{\Pi}_y^{\xi,-\lambda^*}\left(\exp\left\{2\eta\sum_{\ell=r+1}^{r+k}(-\Delta H_\ell)\right\}\right) \\ &= 1 - e^{-2\eta\kappa_0\sqrt{k}}\cdot\prod_{\ell=r+1}^{r+k}\widetilde{\Pi}_y^{\xi,-\lambda^*}\left(\exp\left\{2\eta(-\Delta H_\ell)\right\}\right) \ge 1 - \exp\left\{4(c_2\eta)^2k - 2\eta\kappa_0\sqrt{k}\right\}. \end{aligned}$$

Taking $\eta := \kappa_0/(4c_2^2\sqrt{k}) \le 1/(2c_2)$ and $N(\kappa_0) := (\kappa_0/(2c_2))^2$, then for $k \ge N(\kappa_0)$,

$$\widetilde{\Pi}_y^{\xi,-\lambda^*}\left(\inf_{r<m\le r+k}\sum_{\ell=r+1}^m \Delta H_\ell \ge -\kappa_0\sqrt{k}\right) \ge C_6(\kappa_0) := 1 - \exp\left\{-\kappa_0^2/(4c_2^2)\right\} \in (0,1).$$

The proof is now complete. □

Lemma 3.10. *Let ΔH_k be defined by* (3.29)*. There exist positive constants α^* and C_7, independent of k and r, such that for all $0 \le x \le \alpha^* k$ and $k, r \in \mathbb{Z}^+$,*

$$\inf_{y\in[-1,1]} \widetilde{\Pi}_y^{\xi,-\lambda^*}\left(\sum_{s=r+1}^{r+k} \Delta H_s \ge x\right) \ge C_7 \exp\left\{-\frac{x^2}{C_7 k}\right\}.$$

Proof. We continue the constant label in the proof of Lemma 3.9. We claim that

$$c_3 := \mathrm{essinf}_{\xi,k} \widetilde{\Pi}_y^{\xi,-\lambda^*}\left((\Delta H_k)^2\right) > 0. \tag{3.32}$$

In fact, for $x \in (0,1)$, let

$$\Delta_x H_k := H^1_{k-1+x} - H^1_{k-1} - \widetilde{\Pi}_y^{\xi,-\lambda^*}\left(H^1_{k-1+x} - H^1_{k-1}\right).$$

By the strong Markov property, under $\widetilde{\Pi}_y^{\xi,-\lambda^*}$, $\Delta_x H_k$ and $\Delta H_k - \Delta_x H_k$ are independent. Since ΔH_k and $\Delta_x H_k$ are both centered, we have

$$\begin{aligned}
&\widetilde{\Pi}_y^{\xi,-\lambda^*}\left((\Delta H_k)^2\right) > \widetilde{\Pi}_y^{\xi,-\lambda^*}\left((\Delta_x H_k)^2\right)\\
&= \Pi_{k-1}\left(\exp\left\{\int_0^{H_{k-1+x}} ((m-1)\xi(B_s) - \gamma(-\lambda^*))\,\mathrm{d}s\right\}\frac{\psi(k-1+x,-\lambda^*)}{\psi(k-1,-\lambda^*)}(H_{k-1+x})^2\right)\\
&\quad - \left(\widetilde{\Pi}_y^{\xi,-\lambda^*}\left(H^1_{k-1+x} - H^1_{k-1}\right)\right)^2\\
&\ge \Pi_0\left((H_x)^2 \exp\left\{-\left(\gamma(-\lambda^*) - (m-1)\mathrm{ei}\right) H_x\right\}\right)\\
&\quad - e^{2K_1 x}\left(\Pi_0\left(H_x \exp\left\{-\left(\gamma(-\lambda^*) - (m-1)\mathrm{es}\right) H_x\right\}\right)\right)^2\\
&= x^2\left(\frac{1}{2u_2} e^{-x\sqrt{2u_2}} - \frac{1}{2u_1} e^{-2x(\sqrt{2u_1}-K_1)}\right) + \frac{x}{2\sqrt{2u_2^3}} e^{-x\sqrt{2u_2}} =: \widehat{F}(x),
\end{aligned}$$

where $u_1 := \gamma(-\lambda^*) - (m-1)\mathrm{es} \le \gamma(-\lambda^*) - (m-1)\mathrm{ei} =: u_2$, and in the last equality we used (2.16). Since $\lim_{x\to 0} \frac{\widehat{F}(x)}{x} > 0$, there exists x such that for all $k \in \mathbb{N}$ and all realizations of ξ, $\widetilde{\Pi}_y^{\xi,-\lambda^*}\left((\Delta H_k)^2\right) > \widehat{F}(x) > 0$, which implies the claim holds.

By the sub-exponential property of ΔH_k given by (3.31), there exists a constant $K > 0$, independent of k, such that for all $\ell \ge 1$, $\widetilde{\Pi}_y^{\xi,-\lambda^*}\left(|\Delta H_k|^\ell\right) \le (K\ell)^\ell$ (see [25, Proposition 2.7.1, (ii)]). Therefore, by the trivial inequality that $\ell! \ge (\ell/e)^\ell$, when $|\eta| < 1/(2eK)$,

$$\begin{aligned}
&\widetilde{\Pi}_y^{\xi,-\lambda^*}\left(e^{\eta\Delta H_k}\right) = 1 + \frac{1}{2!}\widetilde{\Pi}_y^{\xi,-\lambda^*}\left((\Delta H_k)^2\right)\eta^2 + \sum_{\ell=3}^\infty \frac{1}{\ell!}\widetilde{\Pi}_y^{\xi,-\lambda^*}\left((\Delta H_k)^\ell\right)|\eta|^\ell\\
&\ge 1 + \frac{c_3}{2}\eta^2 - \sum_{\ell=3}^\infty \frac{1}{(\ell/e)^\ell}(K\ell|\eta|)^\ell = 1 + \frac{c_3}{2}\eta^2 - \sum_{\ell=3}^\infty (eK|\eta|)^\ell\\
&= 1 + \frac{c_3}{2}\eta^2 - \frac{(eK)^3|\eta|^3}{1 - eK|\eta|} \ge 1 + \left(\frac{c_3}{2} - \frac{(eK)^3|\eta|}{2}\right)\eta^2,
\end{aligned}$$

where c_3 is given by (3.32). Now we choose $c_4 > 0$ and c_5 such that $(eK)^3 c_4 = c_3/2$ and that $1 + c_3 c_4^2/4 = e^{c_5 c_4^2}$, then for all $|\eta| \le c_4$, we have

$$1 + \left(\frac{c_3}{2} - \frac{(eK)^3|\eta|}{2}\right)\eta^2 \ge 1 + \frac{c_3}{4}\eta^2 \ge e^{c_5\eta^2},$$

here in the last inequality we used the fact that the function $h(x) := e^{c_5 x} - 1 - c_3 x/4$ is non-positive for $x \in (0, c_4^2)$, which follows easily from our choice of c_5 and c_4. Thus we have

$$\widetilde{\Pi}_y^{\xi,-\lambda^*}\left(e^{\eta\Delta H_k}\right) \ge e^{c_5\eta^2}, \quad |\eta| \le c_4, k \ge 1. \tag{3.33}$$

Now, using (3.31) and (3.33), the desired conclusion follows from [27, Theorem 4] by taking $M = c_4 \wedge (1/2c_2), \alpha = 1, C_1 = (c_2)^2, c_1 = C_5, c'' = M^2, u_1 = u_2 = \cdots = u_k = 1$ and $\alpha^* = c' M\alpha$ with $c' > 0$ being the constant in [27, Theorem 4], and c_2 is the constant in (3.31). □

For $\ell \in \mathbb{Z}$, let

$$\rho_\ell := \frac{\ln \psi(\ell, -\lambda^*) - \ln \psi(\ell-1, -\lambda^*)}{\lambda^* v^*} + \frac{1}{v^*}\left(\frac{\psi_\lambda(\ell, -\lambda^*)}{\psi(\ell, -\lambda^*)} - \frac{\psi_\lambda(\ell-1, -\lambda^*)}{\psi(\ell-1, -\lambda^*)}\right).$$

By (2.22),

$$\rho_\ell = \frac{\ln \phi(\ell, -\lambda^*) - \ln \phi(\ell-1, -\lambda^*)}{\lambda^* v^*} + \frac{1}{v^*}\left(\frac{\phi_\lambda(\ell, -\lambda^*)}{\phi(\ell, -\lambda^*)} - \frac{\phi_\lambda(\ell-1, -\lambda^*)}{\phi(\ell-1, -\lambda^*)}\right). \tag{3.34}$$

Taking logarithm and differentiating with respect to λ in (2.7), and letting $\lambda = -\lambda^*$, we get

$$\begin{aligned}\lambda^* v^* \rho_\ell = &- \ln \Pi_{\ell-1}\left(\exp\left\{\int_0^{H_\ell} ((m-1)\xi(B_s) - \gamma(-\lambda^*))\,\mathrm{d}s\right\}\right) \\ &- \lambda^* v^* \frac{\Pi_{\ell-1}\left(H_\ell \exp\left\{\int_0^{H_\ell} ((m-1)\xi(B_s) - \gamma(-\lambda^*))\,\mathrm{d}s\right\}\right)}{\Pi_{\ell-1}\left(\exp\left\{\int_0^{H_\ell} ((m-1)\xi(B_s) - \gamma(-\lambda^*))\,\mathrm{d}s\right\}\right)}.\end{aligned}$$

Define

$$\begin{aligned}A_1(-\lambda^*) &= \Pi_{\ell-1}\left(\exp\left\{\int_0^{H_\ell} ((m-1)\xi(B_s) - \gamma(-\lambda^*))\,\mathrm{d}s\right\}; \inf_{0\le s\le H_\ell} B_s > j\right), \\ A_2(-\lambda^*) &= \Pi_{\ell-1}\left(\exp\left\{\int_0^{H_\ell} ((m-1)\xi(B_s) - \gamma(-\lambda^*))\,\mathrm{d}s\right\}; \inf_{0\le s\le H_\ell} B_s \le j\right), \\ A_1'(-\lambda^*) &= \Pi_{\ell-1}\left(H_\ell \exp\left\{\int_0^{H_\ell} ((m-1)\xi(B_s) - \gamma(-\lambda^*))\,\mathrm{d}s\right\}; \inf_{0\le s\le H_\ell} B_s > j\right), \\ A_2'(-\lambda^*) &= \Pi_{\ell-1}\left(H_\ell \exp\left\{\int_0^{H_\ell} ((m-1)\xi(B_s) - \gamma(-\lambda^*))\,\mathrm{d}s\right\}; \inf_{0\le s\le H_\ell} B_s \le j\right).\end{aligned}$$

Then

$$\rho_\ell = -\frac{1}{\lambda^* v^*} \ln\left(A_1(-\lambda^*) + A_2(-\lambda^*)\right) - \frac{A_1'(-\lambda^*) + A_2'(-\lambda^*)}{A_1(-\lambda^*) + A_2(-\lambda^*)}.$$

Note that $A_1(-\lambda^*), A_1'(-\lambda^*) \in \mathcal{F}_\ell \cap \mathcal{F}^j$. Using the same notation in (2.24),

$$\begin{aligned}A_1(-\lambda^*) &\le \Pi_{\ell-1}\left(\exp\left\{\left((m-1)\mathrm{es} - \gamma(-\lambda^*)\right) H_\ell\right\}\right) \\ &= \Pi_0\left(\exp\left\{\left((m-1)\mathrm{es} - \gamma(-\lambda^*)\right) H_1\right\}\right) = c_1(-\lambda^*) < 1,\end{aligned}$$

and when $\ell \ge j+2$, we have $j - (\ell - 1) \le -1$ and thus

$$\begin{aligned}A_1(-\lambda^*) &\ge \Pi_{\ell-1}\left(\exp\left\{\left((m-1)\mathrm{ei} - \gamma(-\lambda^*)\right) H_\ell\right\}; \inf_{0\le s\le H_\ell} B_s > j\right) \\ &= \Pi_0\left(\exp\left\{\left((m-1)\mathrm{ei} - \gamma(-\lambda^*)\right) H_1\right\}; \inf_{0\le s\le H_1} B_s > j - (\ell-1)\right) \\ &\ge \Pi_0\left(\exp\left\{\left((m-1)\mathrm{ei} - \gamma(-\lambda^*)\right) H_1\right\}; \inf_{0\le s\le H_1} B_s > -1\right) = c_2(-\lambda^*) > 0.\end{aligned}$$

If $H_\ell \geq \ell - j$, then by **(H2)**,

$$\exp\left\{\int_0^{H_\ell} ((m-1)\xi(B_s) - \gamma(-\lambda^*))\,\mathrm{d}s\right\} \leq \exp\left\{((m-1)\text{es} - \gamma(-\lambda^*))\,(\ell - j)\right\};$$

and if $H_\ell \leq \ell - j$, then

$$\left\{\inf_{0\leq s\leq H_\ell} B_s \leq j\right\} \subset \left\{\inf_{0\leq s\leq \ell-j} B_s \leq j\right\}.$$

Letting $\gamma_0 := \gamma(-\lambda^*) - (m-1)\text{es} > 0$, when $\ell \geq j+2$, by the reflection principle and Markov's inequality,

$$\begin{aligned}
&A_2(-\lambda^*) \leq e^{-\gamma_0(\ell-j)} + \Pi_{\ell-1}\left(\inf_{0\leq s\leq \ell-j} B_s \leq j\right) = e^{-\gamma_0(\ell-j)} + \Pi_0\left(\inf_{0\leq s\leq \ell - j} B_s \leq j-(\ell-1)\right) \\
&= e^{-\gamma_0(\ell-)} + 2\Pi_0\left(B_{\ell-j} \leq j - \ell + 1\right) \\
&\leq e^{-\gamma_0(\ell-j)} + 2\frac{\Pi_0(e^{-B_{\ell-j}/2})}{e^{-(j-\ell+1)/2}} = e^{-\gamma_0(\ell-j)} + 2\frac{e^{(\ell-j)/8}}{e^{-(j-\ell+1)/2}} \leq (1+2e^{1/2})e^{-(\ell-j)\delta}
\end{aligned}$$

with $\delta := \min\{3/8, \gamma_0\}$.

Now we take a constant λ_0 such that $-\rho > \lambda_0 > -\lambda^*$ and $\gamma(-\lambda^*) > \gamma(\lambda_0) > (m-1)\text{es}$. Then there is a constant $c > 0$ such that $\sup_{x>0}\left\{xe^{-(\gamma(-\lambda^*)-\gamma(\lambda_0))x}\right\} \leq c$, and thus

$$\begin{aligned}
0 \leq A_1'(-\lambda^*) &\leq c\Pi_{\ell-1}\left(\exp\left\{\int_0^{H_\ell} ((m-1)\xi(B_s) - \gamma(\lambda_0))\,\mathrm{d}s\right\}; \inf_{0\leq s\leq H_\ell} B_s > j\right) = cA_1(\lambda_0), \\
0 \leq A_2'(-\lambda^*) &\leq c\Pi_{\ell-1}\left(\exp\left\{\int_0^{H_\ell} ((m-1)\xi(B_s) - \gamma(\lambda_0))\,\mathrm{d}s\right\}; \inf_{0\leq s\leq H_\ell} B_s \leq j\right) = cA_2(\lambda_0).
\end{aligned}$$

Noticing that $\gamma(\lambda_0) > (m-1)\text{es}$, using argument similar to those used for A_1 and A_2, with $-\lambda^*$ replaced by λ_0, we have that there exist positive constants $\tilde{\delta}, c_3(\lambda_0), c_4$ such that

$$A_1(\lambda_0) < c_3(\lambda_0), \quad A_2(\lambda_0) \leq c_4 e^{-(\ell-j)\tilde{\delta}}, \quad \forall \ell \geq j+2.$$

Therefore, setting $A_i = A_i(-\lambda^*), A_i' = A_i'(-\lambda^*)$ for simplicity, we conclude that

$$\begin{aligned}
&\left|\rho_\ell + \frac{1}{\lambda^* v^*}\ln(A_1) + \frac{A_1'}{A_1}\right| = \left|\frac{-\ln(A_1+A_2) - \lambda^* v^* \frac{A_1'+A_2'}{A_1+A_2}}{\lambda^* v^*} + \frac{1}{\lambda^* v^*}\ln(A_1) + \frac{A_1'}{A_1}\right| \\
&\leq \frac{1}{\lambda^* v^*}\left|\ln(1 + A_2/A_1)\right| + \frac{A_1 A_2' + A_2 A_1'}{(A_1+A_2)A_1} \\
&\leq \frac{1}{\lambda^* v^*}\frac{A_2}{A_1} + \frac{c_1(-\lambda^*)cc_4 e^{-(\ell-j)\tilde{\delta}} + (1+2e^{1/2})e^{-(\ell-j)\delta}cc_3(\lambda_0)}{c_2(-\lambda^*)^2} \\
&\leq \frac{1}{\lambda^* v^*}\frac{(1+2e^{1/2})e^{-(\ell-j)\delta}}{c_2(-\lambda^*)} + \frac{c_1(-\lambda^*)cc_4 e^{-(\ell-j)\tilde{\delta}} + (1+2e^{1/2})e^{-(\ell-j)\delta}cc_3(\lambda_0)}{c_2(-\lambda^*)^2}.
\end{aligned}$$

Thus there exists some constant $C_8 > 0$ such that for $\forall \ell \geq j+2$,

$$\left|\rho_\ell + \frac{1}{\lambda^* v^*}\ln(A_1) + \frac{A_1'}{A_1}\right| \leq C_8 e^{-(\ell-j)\delta'} \tag{3.35}$$

with $\delta' := \min\{\delta, \tilde{\delta}\} > 0$. Define

$$\overline{\rho}_\ell^{(j)} := -\frac{1}{\lambda^* v^*}\ln(A_1(-\lambda^*)) - \frac{A_1'(-\lambda^*)}{A_1(-\lambda^*)} + C_8 e^{-(\ell-j)\delta'}.$$

Then by (3.35), for $\ell \geq j+2$, we have that

$$\begin{aligned}0 \leq \overline{\rho}_\ell^{(j)} - \rho_\ell =& \left(\overline{\rho}_\ell^{(j)} + \frac{1}{\lambda^* v^*}\ln(A_1(-\lambda^*)) + \frac{A_1'(-\lambda^*)}{A_1(-\lambda^*)}\right) \\ &- \left(\rho_\ell + \frac{1}{\lambda^* v^*}\ln(A_1(-\lambda^*)) + \frac{A_1'(-\lambda^*)}{A_1(-\lambda^*)}\right) \\ \leq& 2C_8 e^{-(\ell-j)\delta'}. \end{aligned} \tag{3.36}$$

Also, $|\overline{\rho}_\ell^{(j)}|$ is uniformly bounded for all $\ell \geq j+2$:

$$\|\overline{\rho}_\ell^{(j)}\|_\infty \leq \frac{1}{\lambda^* v^*}\ln\frac{1}{c_2(-\lambda^*)} + \frac{cc_3(\lambda_0)}{c_2(-\lambda^*)} + C_8 =: M_\rho.$$

By the definitions of $A_1(-\lambda^*)$ and $A_1'(-\lambda^*)$, we know that

$$\overline{\rho}_\ell^{(j)} \in \mathcal{F}_\ell \cap \mathcal{F}^j \quad \text{when} \quad \ell \geq j+2. \tag{3.37}$$

For $q \geq 0$ and $i \geq 1$, let $t_i := 2^i$ and define

$$\begin{aligned} &r_0 := t_{i-1}, \quad L_i := \left\lceil \frac{C_5 t_i^{1/2}}{16 M_\rho} \right\rceil, \\ &s_0 := t_i \wedge \inf\left\{k \geq r_0+1 : \sum_{\ell=r_0+1}^{k} \overline{\rho}_\ell^{(r_0-[L_i/2])} \geq \frac{C_5 t_i^{1/2}}{16}\right\}, \quad r_{q+1} := s_q + L_i, \\ &s_{q+1} := \inf\left\{k \geq r_{q+1}+1 : \sum_{\ell=r_{q+1}+1}^{k} \overline{\rho}_\ell^{(r_{q+1}-[L_i/2])} \geq \frac{C_5 t_i^{1/2}}{16}\right\} \wedge (r_{q+1} + t_i - t_{i-1}), \\ &\mathfrak{K} := \inf\{k : s_k \geq t_i\}, \end{aligned} \tag{3.38}$$

here C_5 is the constant defined in Lemma 3.8.

For $\ell < t_i - t_{i-1}$, define

$$D(\ell) := \left\{\omega : \sum_{r=1}^{k} \overline{\rho}_r^{(-[L_i/2])} < \frac{C_5 t_i^{1/2}}{16}, \forall k < \ell, \quad \sum_{r=1}^{\ell} \overline{\rho}_r^{(-[L_i/2])} \geq \frac{C_5 t_i^{1/2}}{16}\right\}.$$

Also define

$$D(t_i - t_{i-1}) := \left\{\omega : \sum_{r=1}^{\ell} \overline{\rho}_r^{(-[L_i/2])} < \frac{C_5 t_i^{1/2}}{16},\ \forall \ell < t_i - t_{i-1}\right\}.$$

Proposition 3.11. *Fix $i \geq 1$.*

(i) For any $k \geq 1$ and positive integers $\ell_0, ..., \ell_{k-1} < t_i - t_{i-1}, \ell_k \leq t_i - t_{i-1}$,

$$\mathbb{P}(s_j - r_j = \ell_j, 0 \leq j \leq k) \leq \left(1 + \zeta\left(\frac{C_5 t_i^{1/2}}{32 M_\rho}\right)\right)^k \prod_{j=0}^{k} \mathbb{P}(D(\ell_j)). \tag{3.39}$$

(ii) For any $k \geq 1$ and $\ell_0, ..., \ell_k < t_i - t_{i-1}$, when i is sufficiently large so that $t_{i-1} > L_i$,

$$\{\mathfrak{K} = k, s_j - r_j = \ell_j, 0 \leq j \leq k\} \in \mathcal{F}^{t_{i-1}-[L_i/2]} \cap \mathcal{F}_{t_{i+1}}. \tag{3.40}$$

Proof. (i) By induction, we only need to prove that

$$\mathbb{P}(s_j - r_j = \ell_j, 0 \leq j \leq k-1, s_k - r_k = \ell_k)$$

$$\leq \mathbb{P}\left(s_j - r_j = \ell_j, 0 \leq j \leq k-1\right) \mathbb{P}\left(D(\ell_k)\right) \left(1 + \zeta\left(\frac{C_5 t_i^{1/2}}{32 M_\rho}\right)\right). \tag{3.41}$$

For integers r, s such that $r < s$ and $s - r < t_i - t_{i-1}$, define

$$D(r,s) := \left\{\omega : \sum_{\ell=r+1}^{k} \overline{\rho}_\ell^{(r-[L_i/2])} < \frac{C_5 t_i^{1/2}}{16}, \forall k \in [r+1, s), \sum_{\ell=r+1}^{s} \overline{\rho}_\ell^{(r-[L_i/2])} \geq \frac{C_5 t_i^{1/2}}{16}\right\}.$$

Also, when $s - r = t_i - t_{i-1}$, we define

$$D(r,s) := \left\{\omega : \sum_{\ell=r+1}^{k} \overline{\rho}_\ell^{(r-[L_i/2])} < \frac{C_5 t_i^{1/2}}{16}, \forall k \in [r+1, r+t_i - t_{i-1})\right\}.$$

Then if $\ell_j < t_i - t_{i-1}$ for all $0 \leq j \leq k$, it is easy to see that

$$\begin{aligned} &\{s_j - r_j = \ell_j, 0 \leq j \leq k\} \\ =& D(t_{i-1}, t_{i-1} + \ell_0) \cap D(t_{i-1} + \ell_0 + L_i, t_{i-1} + \ell_0 + L_i + \ell_1) \\ &\cap \cdots \cap D(t_{i-1} + kL_i + \ell_0 + ... + \ell_{k-1}, t_{i-1} + kL_i + \ell_0 + ... + \ell_k). \end{aligned} \tag{3.42}$$

It follows from (3.37) that $D(r,s) \in \mathcal{F}_s \cap \mathcal{F}^{r-[L_i/2]}$ when $s - r \leq t_i - t_{i-1}$. Using (3.42), we get

$$\begin{aligned} &\{s_j - r_j = \ell_j, 0 \leq j \leq k-1\} \in \mathcal{F}_{t_{i-1}+(k-1)L_i+\ell_0+...+\ell_{k-1}}, \\ &\{s_k - r_k = \ell_k\} \in \mathcal{F}^{t_{i-1}+kL_i+\ell_0+...+\ell_{k-1}-[L_i/2]}. \end{aligned} \tag{3.43}$$

By the definition of L_i,

$$\begin{aligned} &(t_{i-1} + kL_i + \ell_0 + ... + \ell_{k-1} - [L_i/2]) - (t_{i-1} + (k-1)L_i + \ell_0 + ... + \ell_{k-1}) \\ &= L_i - [L_i/2] \geq \frac{C_5 t_i^{1/2}}{32 M_\rho}, \end{aligned}$$

together with the ζ-mixing condition we conclude that

$$\begin{aligned} &\mathbb{P}\left(s_j - r_j = \ell_j, 0 \leq j \leq k\right) \\ &\leq \mathbb{P}\big(s_j - r_j = \ell_j, 0 \leq j \leq k-1\big) \\ &\quad \times \mathbb{P}\big(D(t_{i-1} + kL_i + \ell_0 + ... + \ell_{k-1}, t_{i-1} + kL_i + \ell_0 + ... + \ell_k)\big) \left(1 + \zeta\left(\frac{C_5 t_i^{1/2}}{32 M_\rho}\right)\right) \\ &= \mathbb{P}\left(s_j - r_j = \ell_j, 0 \leq j \leq k-1\right) \mathbb{P}(D(\ell_k)) \left(1 + \zeta\left(\frac{C_5 t_i^{1/2}}{32 M_\rho}\right)\right), \end{aligned}$$

where in the last equality, we used the property that $\mathbb{P}(D(r,s)) = \mathbb{P}(D(0, s-r)) = \mathbb{P}(D(s-r))$. Thus (3.41) is valid.

(ii) Note that

$$\begin{aligned} &\{\mathfrak{K} = k, s_j - r_j = \ell_j, 0 \leq j \leq k\} = \{s_k \geq t_i, s_{k-1} < t_i, s_j - r_j = \ell_j, 0 \leq j \leq k\} \\ &= \left\{s_j - r_j = \ell_j, 0 \leq j \leq k, s_{k-1} = t_{i-1} + (k-1)L_i + \sum_{j=0}^{k-1} \ell_j < t_i, s_k = t_{i-1} + kL_i + \sum_{j=0}^{k} \ell_j \geq t_i\right\}. \end{aligned}$$

The event in (3.40) is not empty only if $t_{i-1} + (k-1)L_i + \sum_{j=0}^{k-1} \ell_j < t_i$ and $t_{i-1} + kL_i + \sum_{j=0}^{k} \ell_j \geq t_i$. Combining this observation with (3.43), we get that, when $t_{i-1} > L_i$,

$$\{\mathfrak{K} = k, s_j - r_j = \ell_j, 0 \leq j \leq k\} \in \left(\bigcup_{j=0}^{k} \mathcal{F}^{t_{i-1}+jL_i+\ell_0+...+\ell_{j-1}-[L_i/2]}\right) \bigcap \mathcal{F}_{t_{i-1}+kL_i+\ell_0+...+\ell_k}$$

$$\subset \mathcal{F}^{t_{i-1}-[L_i/2]} \cap \mathcal{F}_{L_i+\ell_k+t_i} \subset \mathcal{F}^{t_{i-1}-[L_i/2]} \cap \mathcal{F}_{L_i+(t_i-t_{i-1})+t_i} \subset \mathcal{F}^{t_{i-1}-[L_i/2]} \cap \mathcal{F}_{t_{i+1}}.$$

The proof is now complete. □

Put

$$\alpha_1 := 2\alpha^*/(3C_5), \quad \alpha_2 := \alpha^*/(6C_5), \tag{3.44}$$

where α^* is the constant in Lemma 3.10, and C_5 the constant in Lemma 3.8. Define

$$G_i^1 := \left\{\omega : \max_{r\in[t_{i-1},t_{i+1}],0<k\le t_i^{1/2}/\alpha_1} \sum_{\ell=r+1}^{r+k} \overline{\rho}_\ell^{(t_{i-2})} \le \frac{C_5 t_{i-1}^{1/2}}{32}\right\}.$$

Lemma 3.12. *(i) There exists a constant $C_9 > 0$ such that for sufficiently large i, on G_i^1,*

$$\inf_{y\in[-1,1]} \inf_{x\ge \frac{C_5 t_{i-1}^{1/2}}{2}} \widetilde{\Pi}_y^{\xi,-\lambda^*}\left(\beta_{t_i}^1-\beta_2^1\ge \frac{C_5 t_{i-1}^{1/2}}{2},\ \beta_k^1-\beta_2^1\ge t_i^{1/4}, k=t_{i-1}+1,...,t_i \middle| \beta_{t_{i-1}}^1-\beta_2^1=x\right)$$

$$\ge C_9^{\mathfrak{K}+1}\exp\left\{-\sum_{j=0}^{\mathfrak{K}}\frac{t_i}{C_9(s_j-r_j)}\right\} =: Z_i^1, \tag{3.45}$$

(ii) There exist constants $\varepsilon > 0$ and $K = K(\epsilon)$ such that for i large enough,

$$\mathbb{E}\left(\exp\left\{-\varepsilon 1_{G_i^1}\ln Z_i^1\right\}\right) \le K.$$

Proof. (i) **Step 1** Recall the definition of β_k^1 in (3.13). For $j \ge 0$, define

$$\mathcal{A}_j := \left\{\beta_{s_j}^1 - \beta_2^1 \ge C_5 t_i^{1/2}\right\} \bigcap \bigcap_{k=r_j+1}^{s_j} \left\{\beta_k^1 - \beta_{r_j}^1 + \sum_{\ell=r_j+1}^{k} \rho_\ell \ge -\frac{1}{16}C_5 t_i^{1/2}\right\}$$

$$\widetilde{\mathcal{A}}_j := \bigcap_{k=s_j+1}^{r_{j+1}} \left\{\beta_k^1 - \beta_{s_j}^1 + \sum_{\ell=s_j+1}^{k} \rho_\ell \ge -\frac{1}{16}C_5 t_i^{1/2}\right\}, \quad \mathcal{B} := \left(\bigcap_{j=0}^{\mathfrak{K}} \mathcal{A}_j\right) \bigcap \left(\bigcap_{j=0}^{\mathfrak{K}-1} \widetilde{\mathcal{A}}_j\right).$$

Put

$$\left(Z_i^1\right)^* := \inf_{x\ge \frac{C_5 t_{i-1}^{1/2}}{2}} \widetilde{\Pi}_y^{\xi,-\lambda^*}\left(\beta_{t_i}^1-\beta_2^1\ge \frac{C_5 {t_i}^{1/2}}{2},\ \beta_k^1-\beta_2^1\ge t_i^{1/4}, k=t_{i-1}+1,...,t_i \middle| \beta_{t_{i-1}}^1-\beta_2^1=x\right)$$

$$=\widetilde{\Pi}_y^{\xi,-\lambda^*}\left(\beta_{t_i}^1-\beta_2^1\ge \frac{C_5 {t_i}^{1/2}}{2},\ \beta_k^1-\beta_2^1\ge t_i^{1/4}, k=t_{i-1}+1,...,t_i \middle| \beta_{t_{i-1}}^1-\beta_2^1=\frac{C_5 t_{i-1}^{1/2}}{2}\right),$$

here the last equality follows from the following argument: since $\{\beta_k^1, k \ge 1\}$ has independent increments,

$$\widetilde{\Pi}_y^{\xi,-\lambda^*}\left(\beta_{t_i}^1-\beta_2^1\ge \frac{C_5 {t_i}^{1/2}}{2},\ \beta_k^1-\beta_2^1\ge t_i^{1/4}, k=t_{i-1}+1,...,t_i \middle| \beta_{t_{i-1}}^1-\beta_2^1=x\right)$$

$$=\widetilde{\Pi}_y^{\xi,-\lambda^*}\left(\beta_{t_i}^1-\beta_{t_{i-1}}^1\ge \frac{C_5 {t_i}^{1/2}}{2}-x,\ \beta_k^1-\beta_{t_{i-1}}^1\ge t_i^{1/4}-x, k=t_{i-1}+1,...,t_i \middle| \beta_{t_{i-1}}^1-\beta_2^1=x\right)$$

$$=\widetilde{\Pi}_y^{\xi,-\lambda^*}\left(\beta_{t_i}^1-\beta_{t_{i-1}}^1\ge \frac{C_5 {t_i}^{1/2}}{2}-x,\ \beta_k^1-\beta_{t_{i-1}}^1\ge t_i^{1/4}-x, k=t_{i-1}+1,...,t_i\right)$$

and the last quantity is decreasing in x.

In this step we will show that

$$\left(Z_i^1\right)^* \geq \widetilde{\Pi}_y^{\xi,-\lambda^*}\left(\mathcal{B}\,\middle|\,\beta^1_{t_{i-1}} - \beta^1_2 = \frac{C_5 t_{i-1}^{1/2}}{2}\right). \tag{3.46}$$

Given that $\mathcal{B}$ occurs, under $\widetilde{\Pi}_y^{\xi,-\lambda^*}(\,\cdot\,|\beta^1_{t_{i-1}} - \beta^1_2 = \frac{C_5 t_{i-1}^{1/2}}{2})$, we have, for $0 \leq j \leq \mathfrak{K}$ and $r_j \leq k \leq s_j$,

$$\beta^1_k - \beta^1_{r_j} + \sum_{\ell=r_j+1}^{k} \rho_\ell \geq -\frac{C_5 t_i^{1/2}}{16} \quad \text{on } \mathcal{A}_j.$$

By the definition of s_j we have that for all $r_j < k < s_j$,

$$\sum_{\ell=r_j+1}^{k} \rho_\ell \leq \sum_{\ell=r_j+1}^{k} \overline{\rho}_\ell^{(r_j-[L_i/2])} < \frac{C_5 t_i^{1/2}}{16},$$

and for $k = s_j$,

$$\sum_{\ell=r_j+1}^{s_j} \rho_\ell \leq \sum_{\ell=r_j+1}^{s_j-1} \rho_\ell + M_\rho \leq M_\rho + \frac{C_5 t_i^{1/2}}{16}.$$

Thus, for $0 \leq j \leq \mathfrak{K}$ and $r_j \leq k \leq s_j$, we have

$$\beta^1_k - \beta^1_{r_j} \geq -\frac{C_5 t_i^{1/2}}{8} - M_\rho. \tag{3.47}$$

Note that

$$\beta^1_{r_0} - \beta^1_2 = \beta^1_{t_{i-1}} - \beta^1_2 = \frac{C_5 t_{i-1}^{1/2}}{2} = \frac{C_5 t_i^{1/2}}{2\sqrt{2}}. \tag{3.48}$$

Also for $\mathfrak{K} - 1 \geq j \geq 0$, on $\widetilde{\mathcal{A}}_j$ we have

$$\beta^1_{r_{j+1}} - \beta^1_{s_j} \geq -\sum_{\ell=s_j+1}^{r_{j+1}} \rho_\ell - \frac{C_5 t_i^{1/2}}{16} \geq -\left\lceil \frac{C_5 t_i^{1/2}}{16 M_\rho} \right\rceil M_\rho - \frac{C_5 t_i^{1/2}}{16} \geq -\frac{C_5 t_i^{1/2}}{8} - M_\rho \tag{3.49}$$

and on $\mathcal{A}_j$, $\beta^1_{s_j} - \beta^1_2 \geq C_5 t_i^{1/2}$. Hence, together with (3.49), we have

$$\beta^1_{r_j} - \beta^1_2 = (\beta^1_{r_j} - \beta^1_{s_{j-1}}) + (\beta^1_{s_{j-1}} - \beta^1_2) \geq \frac{7C_5 t_i^{1/2}}{8} - M_\rho, \quad \mathfrak{K} \geq j \geq 1. \tag{3.50}$$

So, combining (3.47), (3.48) and (3.50), we have, for $0 \leq j \leq \mathfrak{K}$ and $r_j \leq k \leq s_j$,

$$\beta^1_k - \beta^1_2 \geq \min\left\{\frac{C_5 t_i^{1/2}}{2\sqrt{2}}, \frac{7C_5 t_i^{1/2}}{8} - M_\rho\right\} - \frac{C_5 t_i^{1/2}}{8} - M_\rho.$$

Let i be sufficiently large so that

$$\min\left\{\frac{C_5 t_i^{1/2}}{2\sqrt{2}}, \frac{7C_5 t_i^{1/2}}{8} - M_\rho\right\} - \frac{C_5 t_i^{1/2}}{8} - M_\rho > t_i^{1/4}.$$

Then $\beta^1_k - \beta^1_2 \geq t_i^{1/4}$ for all $0 \leq j \leq \mathfrak{K}$ and $r_j \leq k \leq s_j$ when i is sufficiently large. Similarly, when $0 \leq j \leq \mathfrak{K} - 1$ and $s_j < k < r_{j+1}$, on $\widetilde{\mathcal{A}}_j$,

$$\beta^1_k - \beta^1_{s_j} \geq -\sum_{\ell=s_j+1}^{k} \rho_\ell - \frac{C_5 t_i^{1/2}}{16} \geq -\sum_{\ell=s_j+1}^{r_{j+1}} M_\rho - \frac{C_5 t_i^{1/2}}{16} \geq -\frac{C_5 t_i^{1/2}}{8} - M_\rho,$$

which implies $\beta^1_k - \beta^1_2 \ge 7C_5 t_i^{1/2}/8 - M_\rho$ since on $\mathcal{A}_j$, $\beta^1_{s_j} - \beta^1_2 \ge C_5 t_i^{1/2}$. For large i such that $7C_5 t_i^{1/2}/8 - M_\rho > t_i^{1/4}$, we also have $\beta^1_k - \beta^1_2 \ge t_i^{1/4}$ holds for all $0 \le j \le \mathfrak{K} - 1$ and $s_j < k < r_{j+1}$. We have also proved that, for all $s_0 \le k \le s_{\mathfrak{K}}$,

$$\beta^1_k - \beta^1_2 \ge \frac{7C_5 t_i^{1/2}}{8} - M_\rho - \frac{C_5 t_i^{1/2}}{8} - M_\rho.$$

Finally, to show $\beta^1_{t_i} - \beta^1_2 \ge C_5 t_i^{1/2}/2$ holds for large i, it suffices to prove that $s_0 \le t_i \le s_{\mathfrak{K}}$, which is trivial by the definitions of s_j and $\mathfrak{K}$,. Thus (3.46) is valid.

Step 2 In this step we will show that there exists a constant $C_9 > 0$ such that

$$\inf_{y\in[-1,1]} \widetilde{\Pi}^{\xi,-\lambda^*}_y \left(\mathcal{B} \,\middle|\, \beta^1_{t_{i-1}} - \beta^1_2 = \frac{C_5 t_{i-1}^{1/2}}{2} \right) \ge C_9^{\mathfrak{K}+1} \exp\left\{ -\sum_{j=0}^{\mathfrak{K}} \frac{t_i}{C_9(s_j - r_j)} \right\}. \tag{3.51}$$

By the strong Markov property,

$$\begin{aligned}
&\widetilde{\Pi}^{\xi,-\lambda^*}_y \left(\mathcal{B} \,\middle|\, \beta^1_{t_{i-1}} - \beta^1_2 = \frac{C_5 t_{i-1}^{1/2}}{2} \right) \\
&\ge \widetilde{\Pi}^{\xi,-\lambda^*}_y \left(\mathcal{A}_0 \,\middle|\, \beta^1_{r_0} - \beta^1_2 = \frac{C_5 t_{i-1}^{1/2}}{2} \right) \cdot \widetilde{\Pi}^{\xi,-\lambda^*}_y \left(\widetilde{\mathcal{A}}_0 \,\middle|\, \beta^1_{s_0} - \beta^1_2 = C_5 t_i^{1/2} \right) \\
&\quad \times \widetilde{\Pi}^{\xi,-\lambda^*}_y \left(\mathcal{A}_1 \,\middle|\, \beta^1_{r_1} - \beta^1_2 = \frac{7C_5 t_i^{1/2}}{8} - M_\rho \right) \times \ldots \times \widetilde{\Pi}^{\xi,-\lambda^*}_y \left(\mathcal{A}_{\mathfrak{K}} \,\middle|\, \beta^1_{r_{\mathfrak{K}}} - \beta^1_2 = \frac{7C_5 t_i^{1/2}}{8} - M_\rho \right) \\
&= \widetilde{\Pi}^{\xi,-\lambda^*}_y \left(\mathcal{A}_0 \,\middle|\, \beta^1_{r_0} - \beta^1_2 = \frac{C_5 t_{i-1}^{1/2}}{2} \right) \prod_{j=0}^{\mathfrak{K}-1} \widetilde{\Pi}^{\xi,-\lambda^*}_y (\widetilde{\mathcal{A}}_j) \prod_{j=1}^{\mathfrak{K}} \left(\mathcal{A}_j \,\middle|\, \beta^1_{r_j} - \beta^1_2 = \frac{7C_5 t_i^{1/2}}{8} - M_\rho \right).
\end{aligned} \tag{3.52}$$

Note that $\widetilde{\mathcal{A}}_j$ can be rewritten as

$$\widetilde{\mathcal{A}}_j := \left\{ \inf_{s_j < k \le r_{j+1}} \left(H^1_k - H^1_{s_j} - \widetilde{\Pi}^{\xi,-\lambda^*}_y \left(H^1_k - H^1_{s_j} \right) \right) \ge -\frac{C_5 t_i^{1/2}}{16} \right\}.$$

This is because

$$\begin{aligned}
&\beta^1_k - \beta^1_{s_j} + \sum_{\ell=s_j+1}^{k} \rho_\ell = \sum_{\ell=s_j+1}^{k} \left(\beta^1_\ell - \beta^1_{\ell-1} + \rho_\ell \right) \\
&= \sum_{\ell=s_j+1}^{k} \left(H^1_\ell - H^1_{\ell-1} + \frac{1}{v^*} \left(\frac{\psi_\lambda(\ell,-\lambda^*)}{\psi(\ell,-\lambda^*)} - \frac{\psi_\lambda(\ell-1,-\lambda^*)}{\psi(\ell-1,-\lambda^*)} \right) \right) \\
&= \sum_{\ell=s_j+1}^{k} \left(H^1_\ell - H^1_{\ell-1} - \frac{\Pi_{\ell-1}\left(H_\ell \exp\left\{ \int_0^{H_\ell} ((m-1)\xi(B_s) - \gamma(-\lambda^*))\,\mathrm{d}s \right\} \right)}{\Pi_{\ell-1}\left(\exp\left\{ \int_0^{H_\ell} ((m-1)\xi(B_s) - \gamma(-\lambda^*))\,\mathrm{d}s \right\} \right)} \right) \\
&= \sum_{\ell=s_j+1}^{k} \left(H^1_\ell - H^1_{\ell-1} - \widetilde{\Pi}^{\xi,-\lambda^*}_{\ell-1} \left(H^1_\ell \right) \right) = \sum_{\ell=s_j+1}^{k} \left(H^1_\ell - H^1_{\ell-1} - \widetilde{\Pi}^{\xi,-\lambda^*}_y \left(H^1_\ell - H^1_{\ell-1} \right) \right).
\end{aligned}$$

Also note that $r_{j+1} - s_j < C_5 t_i^{1/2}/(16M_\rho) + 1 < t_i$ for sufficiently large i and $r_{j+1} - s_j \ge C_5 t_i^{1/2}/(16M_\rho)$. Taking $\kappa_0 = C_5/16$ in Lemma 3.9, $C_6 := C_6(\kappa_0) \in (0,1)$ with $\kappa_0 = C_5/16$ such that for large i,

$$\widetilde{\Pi}^{\xi,-\lambda^*}_y \left(\widetilde{\mathcal{A}}_j \right) \ge C_6. \tag{3.53}$$

Define $d_0 = C_5 t_{i-1}^{1/2}/2$ and $d_j = 7C_5 t_i^{1/2}/8 - M_\rho$ for $j \geq 1$. Let $f(X) = x_1 + ... + x_{s_j - r_j}, g(X) = \inf_{1\leq k\leq s_j-r_j}(x_1 + ... + x_k)$ and $X = (x_1, ..., x_{s_j-r_j})$. By Harris' inequality, see [5, Theorem 2.15], we have for all $j \geq 0$,

$$\begin{aligned}
&\widetilde{\Pi}_y^{\xi,-\lambda^*}\left(\mathcal{A}_j \middle| \beta^1_{r_j} - \beta^1_2 = d_j\right) \geq \widetilde{\Pi}_y^{\xi,-\lambda^*}\left(\beta^1_{s_j} - \beta^1_2 \geq C_5 t_i^{1/2} \middle| \beta^1_{r_j} - \beta^1_2 = d_j\right)\\
&\quad \times \widetilde{\Pi}_y^{\xi,-\lambda^*}\left(\inf_{r_j<k\leq s_j}\left(H^1_k - H^1_{r_j} - \widetilde{\Pi}_y^{\xi,-\lambda^*}\left(H^1_k - H^1_{r_j}\right)\right) \geq -\frac{C_5 t_i^{1/2}}{16}\right)\\
&= \widetilde{\Pi}_y^{\xi,-\lambda^*}\left(\beta^1_{s_j} - \beta^1_{r_j} \geq C_5 t_i^{1/2} - d_j\right)\\
&\quad \times \widetilde{\Pi}_y^{\xi,-\lambda^*}\left(\inf_{r_j<k\leq s_j}\left(H^1_k - H^1_{r_j} - \widetilde{\Pi}_y^{\xi,-\lambda^*}\left(H^1_k - H^1_{r_j}\right)\right) \geq -\frac{C_5 t_i^{1/2}}{16}\right)\\
&\geq \widetilde{\Pi}_y^{\xi,-\lambda^*}\left(\beta^1_{s_j} - \beta^1_{r_j} \geq C_5 t_i^{1/2}\right)\\
&\quad \times \widetilde{\Pi}_y^{\xi,-\lambda^*}\left(\inf_{r_j<k\leq s_j}\left(H^1_k - H^1_{r_j} - \widetilde{\Pi}_y^{\xi,-\lambda^*}\left(H^1_k - H^1_{r_j}\right)\right) \geq -\frac{C_5 t_i^{1/2}}{16}\right)\\
&=: (I) \times (II). \qquad (3.54)
\end{aligned}$$

We treat (I) first. Note that $C_5 t_i^{1/2}/16 + M_\rho \leq C_5 t_i^{1/2}/2$ for large i. By the definition of s_j, we have $\sum_{\ell=r_j+1}^{s_j} \rho_\ell \leq C_5 t_i^{1/2}/2$ and

$$\begin{aligned}
(I) =&\widetilde{\Pi}_y^{\xi,-\lambda^*}\left(\beta^1_{s_j} - \beta^1_{r_j} + \sum_{\ell=r_j+1}^{s_j}\rho_\ell \geq C_5 t_i^{1/2} + \sum_{\ell=r_j+1}^{s_j}\rho_\ell\right)\\
\geq&\widetilde{\Pi}_y^{\xi,-\lambda^*}\left(H^1_{s_j} - H^1_{r_j} - \widetilde{\Pi}_y^{\xi,-\lambda^*}\left(H^1_{s_j} - H^1_{r_j}\right) \geq \frac{3C_5 t_i^{1/2}}{2}\right). \qquad (3.55)
\end{aligned}$$

By the definitions of s_j and r_j, we have $t_i \geq s_j - r_j \geq C_5 t_i^{1/2}/(16M_\rho)$. By (3.36) and the definition of s_j, for large i such that $2C_8 t_i e^{-C_5 t_i^{1/2}\delta'/(32M_\rho)} < C_5 t_i^{1/2}/32$, when $s_j - r_j < t_i - t_{i-1}$,

$$\begin{aligned}
\frac{C_5 t_i^{1/2}}{16} &\leq \sum_{\ell=r_j+1}^{s_j} \overline{\rho}_\ell^{(r_j-[L_i/2])} \leq \sum_{\ell=r_j+1}^{s_j}\rho_\ell + 2C_8(s_j - r_j)\exp\left\{-\left(r_j + 1 - (r_j - [L_i/2])\right)\delta'\right\}\\
&\leq \sum_{\ell=r_j+1}^{s_j}\rho_\ell + 2C_8 t_i e^{-([L_i/2]+1)\delta'} \leq \sum_{\ell=r_j+1}^{s_j}\rho_\ell + 2C_8 t_i e^{-C_5 t_i^{1/2}\delta'/(32M_\rho)}\\
&\leq \sum_{\ell=r_j+1}^{s_j}\rho_\ell + \frac{C_5 t_i^{1/2}}{32} \leq \sum_{\ell=r_j+1}^{s_j}\overline{\rho}_\ell^{(t_{i-2})} + \frac{C_5 t_i^{1/2}}{32}.
\end{aligned}$$

This implies that, on G_i^1, $s_j - r_j \geq t_i^{1/2}/\alpha_1$. When $s_j - r_j = t_i - t_{i-1}$, we can also take i large so that $s_j - r_j = t_i - t_{i-1} > t_i^{1/2}/\alpha_1$. By the definitions of α_1 (see (3.44)) and α^*, we have $\alpha^*(s_j - r_j) \geq 3C_5 t_i^{1/2}/2$ and hence by Lemma 3.10 (with $k = s_j - r_j$ and $x = 3C_5 t_i^{1/2}/2$) and (3.55), the constant C_7 defined in Lemma 3.10 satisfies that on G_i^1 and for any j,

$$(I) \geq C_7 \exp\left\{-\frac{9C_5^2 t_i}{4C_7(s_j - r_j)}\right\}. \qquad (3.56)$$

Now we treat (II). Taking $\kappa_0 = C_5/16$ in Lemma 3.9, we get

$$(II) = \widetilde{\Pi}_y^{\xi,-\lambda^*}\left(\inf_{r_j<k\leq s_j}\left(H^1_k - H^1_{r_j} - \widetilde{\Pi}_y^{\xi,-\lambda^*}\left(H^1_k - H^1_{r_j}\right)\right) \geq -\frac{C_5 t_i^{1/2}}{16}\right) \geq C_6, \qquad (3.57)$$

where $C_6 = C_6(\kappa_0)$ with $\kappa_0 = C_5/16$. If $s_j = r_j + t_i - t_{i-1}$, then $s_j - r_j \geq C_5 t_i^{1/2}/(16M_\rho)$ for large i and (3.57) also holds. Plugging (3.56) and (3.57) into (3.54), we obtain

$$\widetilde{\Pi}_y^{\xi,-\lambda^*}\left(\mathcal{A}_j \middle| \beta^1_{r_j} - \beta^1_2 = d_j\right) \geq C_7 \cdot C_6 \exp\left\{-\frac{9C_5^2 t_i}{4C_7(s_j - r_j)}\right\}. \tag{3.58}$$

Combining (3.52), (3.53) and (3.58), we conclude that on G_i^1,

$$\begin{aligned} Z_i^1 &\geq C_7 \cdot C_6 (C_7 \cdot (C_6)^2)^{\mathfrak{K}} \exp\left\{-\sum_{j=0}^{\mathfrak{K}} \frac{9C_5^2 t_i}{4C_7(s_j - r_j)}\right\} \\ &\geq (C_7 \cdot (C_6)^2)^{\mathfrak{K}+1} \exp\left\{-\sum_{j=0}^{\mathfrak{K}} \frac{9C_5^2 t_i}{4C_7(s_j - r_j)}\right\}. \end{aligned}$$

Since $\mathfrak{K}$ and $s_j - r_j$ does not depend on $y \in [-1, 1]$, (3.51) holds with $C_9 := \left(C_7 \cdot C_6^2\right) \wedge \left(4C_7/(9C_5^2)\right)$.

Combining Step 1 and Step 2, we get (3.45).

(ii) Define

$$\mathfrak{J} := \inf\{j : s_j - r_j \geq t_i - t_{i-1}\}.$$

Then $\mathfrak{K} \leq \mathfrak{J}$. In fact, on the event $\mathfrak{K} = k$, we have $s_k \geq t_i$ but $s_{k-1} < t_i$, and then for all $0 \leq j < k$, $s_j - r_j \leq s_{k-1} - t_{i-1} < t_i - t_{i-1}$, which implies that $\mathfrak{J} \geq k$. For $\varepsilon \in (0, 1]$, by (3.45),

$$\begin{aligned} &\mathbb{E}\left(\exp\left\{-\varepsilon 1_{G_i^1} \ln Z_i^1\right\}\right) \leq \mathbb{E}\left(\exp\left\{\varepsilon 1_{G_i^1}\left(-(\mathfrak{J}+1)\ln C_9 + \sum_{j=0}^{\mathfrak{J}} \frac{t_i}{C_9(s_j - r_j)}\right)\right\}\right) \\ &\leq \mathbb{E}\left(\exp\left\{\varepsilon\left(-(\mathfrak{J}+1)\ln C_9 + \sum_{j=0}^{\mathfrak{J}} \frac{t_i}{C_9(s_j - r_j)}\right)\right\}\right) \\ &\leq \sum_{k=0}^{\infty} \left(\frac{1}{C_9}\right)^{\varepsilon(k+1)} \mathbb{E}\left(\exp\left\{\varepsilon \frac{t_i}{C_9(s_k - r_k)} + \varepsilon \sum_{j=0}^{k-1} \frac{t_i}{C_9(s_j - r_j)}\right\} 1_{\{\mathfrak{J}=k\}}\right). \end{aligned}$$

By (3.39),

$$\begin{aligned} &\mathbb{E}\left(\exp\left\{\varepsilon \frac{t_i}{C_9(s_k - r_k)} + \varepsilon \sum_{j=0}^{k-1} \frac{t_i}{C_9(s_j - r_j)}\right\} 1_{\{\mathfrak{J}=k\}}\right) \\ &= \mathbb{E}\Bigg(\exp\left\{\varepsilon \frac{t_i}{C_9(s_k - r_k)}\right\} 1_{\{s_k - r_k = t_i - t_{i-1}\}} \prod_{j=0}^{k-1} \exp\left\{\varepsilon \frac{t_i}{C_9(s_j - r_j)}\right\} 1_{\{s_j - r_j < t_i - t_{i-1}\}}\Bigg) \\ &\leq e^{2\varepsilon/C_9} \sum_{1 \leq \ell_0, \ldots, \ell_{k-1} < t_i - t_{i-1}} \prod_{j=0}^{k-1} \exp\left\{\varepsilon \frac{t_i}{C_9 \ell_j}\right\} \mathbb{P}\left(s_j - r_j = \ell_j, 0 \leq j \leq k-1\right) \\ &\leq \left(1 + \zeta\left(\frac{C_5 t_i^{1/2}}{32M_\rho}\right)\right)^k e^{2\varepsilon/C_9} \sum_{1 \leq \ell_0, \ldots, \ell_{k-1} < t_i - t_{i-1}} \prod_{j=0}^{k-1} \left(\exp\left\{\varepsilon \frac{t_i}{C_9 \ell_j}\right\} \mathbb{P}\left(D(\ell_j)\right)\right) \\ &= e^{2\varepsilon/C_9} \left(1 + \zeta\left(\frac{C_5 t_i^{1/2}}{32M_\rho}\right)\right)^k \left(\sum_{\ell=1}^{t_i - t_{i-1} - 1} \exp\left\{\varepsilon \frac{t_i}{C_9 \ell}\right\} \mathbb{P}\left(D(\ell)\right)\right)^k. \end{aligned}$$

Therefore,

$$\mathbb{E}\left(\exp\left\{-\varepsilon 1_{G_i^1} \ln Z_i^1\right\}\right)$$

$$\leq \sum_{k=0}^{\infty} \frac{e^{2\varepsilon/C_9}}{(C_9)^{\varepsilon(k+1)}} \left(1 + \zeta\left(\frac{C_5 t_i^{1/2}}{32M_\rho}\right)\right)^k \left(\sum_{\ell=1}^{t_i - t_{i-1}-1} \exp\left\{\varepsilon \frac{t_i}{C_9 \ell}\right\} \mathbb{P}\left(D(\ell)\right)\right)^k. \tag{3.59}$$

Note that $t_i - t_{i-1} = t_i/2$, and $D(\ell)$ are disjoint for all $\ell < t_i/2$. Therefore, for all $q < t_i/2$,

$$\sum_{\ell=1}^{q} \mathbb{P}(D(\ell)) = \mathbb{P}\left(\bigcup_{\ell=1}^{q} D(\ell)\right) = \mathbb{P}\left(\max_{1\leq k\leq q} \sum_{\ell=1}^{k} \overline{\rho}_\ell^{(-[L_i/2])} \geq \frac{C_5 t_i^{1/2}}{16}\right). \tag{3.60}$$

Using (3.36), for any $q < t_i/2$ and large i such that $C_5 t_i^{1/2}/16 - 2C_8 e^{-([L_i/2]+1)\delta'} \cdot t_i/2 \geq C_5 t_i^{1/2}/32$, we have

$$\begin{aligned}
&\mathbb{P}\left(\max_{1\leq k\leq q} \sum_{\ell=1}^{k} \overline{\rho}_\ell^{(-[L_i/2])} \geq \frac{C_5 t_i^{1/2}}{16}\right) \leq \mathbb{P}\left(\max_{1\leq k\leq q} \sum_{\ell=1}^{k} \rho_\ell \geq \frac{C_5 t_i^{1/2}}{16} - 2C_8 e^{-(1+[L_i/2])\delta'} \cdot q\right) \\
&\leq \mathbb{P}\left(\max_{1\leq k\leq q} \sum_{\ell=1}^{k} \rho_\ell \geq \frac{C_5 t_i^{1/2}}{32}\right).
\end{aligned} \tag{3.61}$$

Recall that $\rho_\ell \in \mathcal{F}_\ell$ and that $\mathbb{E}(\rho_\ell) = 0$, so for all $k \leq \ell$, by ζ-mixing, we have that $\left|\mathbb{E}\left(\rho_k | \mathcal{F}^\ell\right)\right| \leq \mathbb{E}(|\rho_k|)\zeta(\ell - k)$ and therefore, for all $\ell \geq 1$,

$$\begin{aligned}
&\sup_{1\leq j\leq \ell} \left(\left\|\rho_j^2\right\|_\infty + 2\left\|\rho_\ell \sum_{k=j}^{\ell-1} \mathbb{E}\left(\rho_k | \mathcal{F}^\ell\right)\right\|_\infty\right) \\
&\leq M_\rho^2 + \sup_{1\leq j\leq \ell} 2\|\rho_\ell\|_\infty \sum_{k=j}^{\ell-1} \mathbb{E}(|\rho_k|)\zeta(\ell-k) \leq M_\rho^2 + 2M_\rho^2 \sum_{k=0}^{\infty} \zeta(k) < \infty.
\end{aligned}$$

By [23, Theorem 2.4], or more generally, [14, Theorem 1], there exists a constant c_1 such that for any $x > 0$ and $q \in \mathbb{Z}^+$,

$$\mathbb{P}\left(\max_{1\leq k\leq q} \sum_{\ell=1}^{k} \rho_\ell > x\right) \leq c_1 e^{-x^2/(c_1 q)}. \tag{3.62}$$

Using (3.60), (3.61) and (3.62), we conclude that, when i is large enough, for all $q < t_i/2$,

$$\sum_{\ell=1}^{q} \mathbb{P}(D(\ell)) \leq \mathbb{P}\left(\max_{1\leq k\leq q} \sum_{\ell=1}^{k} \rho_\ell \geq \frac{C_5 t_i^{1/2}}{32}\right) \leq c_1 e^{-C_5^2 t_i/(32^2 c_1 q)} =: c_1 e^{-c_2 t_i/q}. \tag{3.63}$$

Also, for $q = t_i/2 - 1$, we have that

$$\sum_{\ell=1}^{t_i/2-1} \mathbb{P}(D(\ell)) \leq \mathbb{P}\left(\max_{1\leq k\leq t_i/2} \sum_{\ell=1}^{k} \rho_\ell \geq \frac{C_5 t_i^{1/2}}{32}\right).$$

Recall the definition of ρ_l given by (3.34). If $\sigma_{-\lambda^*}^2 > 0$, then by Lemma 2.6 and the continuous mapping theorem,

$$\mathbb{P}\left(\max_{1\leq k\leq t_i/2} \sum_{\ell=1}^{k} \rho_\ell \geq \frac{C_5 t_i^{1/2}}{32}\right) \overset{i\to\infty}{\longrightarrow} \Pi_0\left(\sup_{0\leq t\leq 1/2} \sigma_{-\lambda^*} B_t/(\lambda^* v^*) \geq \frac{C_5}{32}\right) < 1.$$

Otherwise, if $\sigma_{-\lambda^*}^2 = 0$, by Lemma 2.6, when $C_5 t_i^{1/2}/32 > \sup_{k\in\mathbb{Z}_+} \|\sum_{\ell=1}^{k} \rho_\ell\|_\infty$, we have

$$\mathbb{P}\left(\max_{1\leq k\leq t_i/2} \sum_{\ell=1}^{k} \rho_\ell \geq \frac{C_5 t_i^{1/2}}{32}\right) = 0.$$

In conclusion, there exists a constant $c_3 \in (0,1)$ such that for large i,

$$\sum_{\ell=1}^{t_i/2-1} \mathbb{P}(D(\ell)) \le \mathbb{P}\left(\max_{1\le k\le t_i/2}\sum_{\ell=1}^{k}\rho_\ell \ge \frac{C_5 t_i^{1/2}}{32}\right) \le c_3. \tag{3.64}$$

For i so that $t_i < 4(t_i/2-1)$, by (3.63) and (3.64), applying Abel's equation $\sum_{k=1}^{q} a_k b_k = \sum_{k=1}^{q-1}(a_k - a_{k+1})\sum_{\ell=1}^{k} b_\ell + a_q \sum_{\ell=1}^{q} b_\ell$, we get

$$\begin{aligned}
&\sum_{\ell=1}^{t_i-t_{i-1}-1} \exp\left\{\varepsilon\frac{t_i}{C_9\ell}\right\}\mathbb{P}\left(D(\ell)\right)\\
&= \sum_{\ell=1}^{t_i/2-2}\left(\exp\left\{\varepsilon\frac{t_i}{C_9\ell}\right\} - \exp\left\{\varepsilon\frac{t_i}{C_9(\ell+1)}\right\}\right)\sum_{q=1}^{\ell}\mathbb{P}(D(q))\\
&\quad + \exp\left\{\varepsilon\frac{t_i}{C_9(t_i/2-1)}\right\}\sum_{\ell=1}^{t_i/2-1}\mathbb{P}(D(\ell))\\
&\le \sum_{\ell=1}^{t_i/2-2}\left(\exp\left\{\varepsilon\frac{t_i}{C_9\ell}\right\} - \exp\left\{\varepsilon\frac{t_i}{C_9(\ell+1)}\right\}\right)c_1 e^{-c_2 t_i/\ell} + c_3\exp\left\{\varepsilon\frac{4}{C_9}\right\}.
\end{aligned} \tag{3.65}$$

Noting that $(e^{a/x})' = -ax^{-2}e^{a/x}$, we have

$$\begin{aligned}
&\left(\exp\left\{\varepsilon\frac{t_i}{C_9\ell}\right\} - \exp\left\{\varepsilon\frac{t_i}{C_9(\ell+1)}\right\}\right)c_1 e^{-c_2 t_i/\ell}\\
&= c_1 e^{-c_2 t_i/\ell}\int_\ell^{\ell+1}\frac{\varepsilon t_i}{C_9 x^2}e^{\varepsilon\frac{t_i}{C_9 x}}\mathrm{d}x \le c_1\int_\ell^{\ell+1}\frac{\varepsilon t_i}{C_9 x^2}e^{\varepsilon\frac{t_i}{C_9 x}}e^{-c_2 t_i/x}\mathrm{d}x.
\end{aligned}$$

So by (3.65), we conclude that

$$\begin{aligned}
&\sum_{\ell=1}^{t_i-t_{i-1}-1}\exp\left\{\varepsilon\frac{t_i}{C_9\ell}\right\}\mathbb{P}\left(D(\ell)\right) \le c_1\int_1^{t_i/2-1}\frac{\varepsilon t_i}{C_9x^2}e^{\varepsilon\frac{t_i}{C_9x}}e^{-c_2t_i/x}\mathrm{d}x + c_3\exp\left\{\varepsilon\frac{4}{C_9}\right\}\\
&\overset{z:=t_i/x}{=} c_1\frac{\varepsilon}{C_9}\int_{t_i/(t_i/2-1)}^{t_i} e^{z(\varepsilon/C_9-c_2)}\mathrm{d}z + c_3\exp\left\{\varepsilon\frac{4}{C_9}\right\}\\
&\le c_1\frac{\varepsilon}{C_9}\int_2^\infty e^{z(\varepsilon/C_9-c_2)}\mathrm{d}z + c_3\exp\left\{\varepsilon\frac{4}{C_9}\right\} =: F(\varepsilon).
\end{aligned} \tag{3.66}$$

Since $F(0) < 1$, we may take i_1 sufficiently large and ε sufficiently small such that

$$F(\varepsilon)\cdot\frac{1+\zeta\left(C_5 t_{i_1}^{1/2}/(32M_\rho)\right)}{(C_9)^\varepsilon} < 1.$$

Since ζ is decreasing, for sufficiently small ε and $i \ge i_1$, using (3.59) and (3.66), we get

$$\begin{aligned}
&\mathbb{E}\left(\exp\left\{-\varepsilon 1_{G_i^1}\ln Z_i^1\right\}\right)\\
&\le \frac{e^{2\varepsilon/C_9}}{(C_9)^\varepsilon}\sum_{k=0}^\infty\left(\frac{1+\zeta\left(C_5t_i^{1/2}/(32M_\rho)\right)}{(C_9)^\varepsilon}\right)^k\left(\sum_{\ell=1}^{t_i-t_{i-1}-1}\exp\left\{\varepsilon\frac{t_i}{C_9\ell}\right\}\mathbb{P}\left(D(\ell)\right)\right)^k\\
&\le \frac{e^{2\varepsilon/C_9}}{(C_9)^\varepsilon}\sum_{k=0}^\infty\left(\frac{1+\zeta\left(C_5t_i^{1/2}/(32M_\rho)\right)}{(C_9)^\varepsilon}\cdot F(\varepsilon)\right)^k
\end{aligned}$$

$$\le \frac{e^{2\varepsilon/C_9}}{(C_9)^\varepsilon}\sum_{k=0}^{\infty}\left(\frac{1+\zeta\left(C_5 t_{i_1}^{1/2}/(32M_\rho)\right)}{(C_9)^\varepsilon}\cdot F(\varepsilon)\right)^k =: K<\infty. \qquad \square$$

Using similar arguments, we have the following lemmas 3.13-3.15. Since the arguments are similar, we will only sketch the proofs.

In the following, $\bar{C}$ is a positive constant which will be specified later in the proof of Lemma 3.3. Put

$$r_0^1 := [\bar{C}\ln n], \qquad L_n^1 := \left\lceil \frac{C_5\bar{C}\ln n}{4M_\rho}\right\rceil, \qquad k(n) := \left[\log_2\left([\bar{C}\ln n]^2\right)\right],$$

$$s_0^1 := t_{k(n)} \wedge \inf\left\{k\ge r_0^1+1 : \sum_{\ell=r_0^1+1}^{k}\overline{\rho}_\ell^{(r_0^1-[L_n^1/2])} \ge \frac14 C_5\bar{C}\ln n\right\}, \quad r_{j+1}^1 := s_j^1+L_n^1,$$

$$s_{j+1}^1 := \inf\left\{k\ge r_{j+1}^1+1 : \sum_{\ell=r_{j+1}^1+1}^{k}\overline{\rho}_\ell^{(r_{j+1}^1-[L_n^1/2])} \ge \frac14 C_5\bar{C}\ln n\right\}\wedge\left(r_{j+1}^1+t_{k(n)}-[\bar{C}\ln n]\right),$$

$$\mathfrak{K}^1 := \inf\{k : s_k^1\ge t_{k(n)}\}.$$

Define

$$\widetilde{G}_n^1 := \left\{\max_{[\bar{C}\ln n]\le r\le 2t_{k(n)},0<k\le[\bar{C}\ln n]/\alpha_2}\ \sum_{m=r+1}^{r+k}\overline{\rho}_m^{\left([2^{-1}\bar{C}\ln n]\right)} \le \frac18 C_5\bar{C}\ln n\right\},$$

where α_2 is defined in (3.44). Recall that C_9 is the constant in Lemma 3.12.

Lemma 3.13. *(i) When n is large enough, on $\widetilde{G}_n^1$, we have that*

$$\inf_{y\in[-1,1]}\inf_{x\ge C_5[\bar{C}\ln n]}\widetilde{\Pi}_y^{\xi,-\lambda^*}\Big\{\beta_k^1-\beta_2^1\ge 0,\ \forall k \text{ such that } [\bar{C}\ln n]+1\le k\le t_{k(n)},$$
$$\beta_{t_{k(n)}}^1-\beta_2^1\ge\frac{C_5 t_{k(n)}^{1/2}}{2}\ \Big|\ \beta_{[\bar{C}\ln n]}^1-\beta_2^1=x\Big\}$$
$$\ge C_9^{\mathfrak{K}^1+1}\exp\left\{-\sum_{j=0}^{\mathfrak{K}^1}\frac{t_{k(n)}}{C_9(s_j^1-r_j^1)}\right\} =: Y_n^1. \tag{3.67}$$

(ii) There exist positive constants ε and $K=K(\varepsilon)$ such that for n large enough,

$$\mathbb{E}\left(\exp\left\{-\varepsilon 1_{\widetilde{G}_n^1}\ln Y_n^1\right\}\right)\le K.$$

Proof. For $\ell<t_{k(n)}-[\bar{C}\ln n]$, define

$$D^1(\ell) := \left\{\omega : \sum_{r=1}^{k}\overline{\rho}_r^{(-[L_n^1/2])}<\frac14 C_5\bar{C}\ln n,\ \forall k\in[1,\ell),\ \sum_{r=1}^{\ell}\overline{\rho}_r^{(-[L_n^1/2])}\ge\frac14 C_5\bar{C}\ln n\right\}.$$

Using the ζ-mixing condition, similar to (3.41), we have

$$\mathbb{P}\left(s_j^1-r_j^1=\ell_j, 0\le j\le k-1, s_k-r_k=\ell_k\right)$$
$$\le \mathbb{P}\left(s_j^1-r_j^1=\ell_j,0\le j\le k-1\right)\mathbb{P}\left(D^1(\ell_k)\right)\left(1+\zeta\left(\frac{C_5\bar{C}\ln n}{8M_\rho}\right)\right). \tag{3.68}$$

As variants of $\mathcal{A}_j$, $(\mathcal{A}_j)'$ and $\mathcal{B}$ in Lemma 3.12, we define

$$\mathcal{A}_j^1 := \left\{\beta_{s_j^1}^1-\beta_2^1\ge 4C_5\bar{C}\ln n\right\}\bigcap\bigcap_{k=r_j^1+1}^{s_j^1}\left\{\beta_k^1-\beta_{r_j^1}^1+\sum_{\ell=r_j^1+1}^{k}\rho_\ell\ge-\frac14 C_5\bar{C}\ln n\right\},$$

$$(\mathcal{A}_j^1)' := \bigcap_{k=s_j^1+1}^{r_{j+1}^1} \left\{ \beta_k^1 - \beta_{s_j^1}^1 + \sum_{\ell=s_j^1+1}^{k} \rho_\ell \geq -\frac{1}{4} C_5 \bar{C} \ln n \right\}, \quad \mathcal{B}^1 := \left(\bigcap_{j=0}^{\mathfrak{K}^1} \mathcal{A}_j^1 \right) \bigcap \left(\bigcap_{j=0}^{\mathfrak{K}^1-1} (\mathcal{A}_j^1)' \right).$$

Then repeating the arguments in Step 1 and Step 2 in the proof of Lemma 3.12, also note that $C_5[\bar{C}\ln n]/2 \geq C_5 t_{k(n)}^{1/2}/2$, we get (3.67).

Define

$$\mathfrak{J}^1 := \inf\{j : s_j^1 - r_j^1 \geq t_{k(n)} - [\bar{C}\ln n]\},$$

and

$$b_n := t_{k(n)} - [\bar{C}\ln n].$$

By repeating the argument of the proof of Lemma 3.12 (ii), we get that when n is large enough so that $t_{k(n)} \leq 2\left(t_{k(n)} - [\bar{C}\ln n]\right)$,

$$\begin{aligned}
&\mathbb{E}\left(\exp\left\{-\varepsilon 1_{\tilde{G}_n^1} \ln Y_n^1\right\}\right) \\
&\leq \sum_{k=0}^{\infty} \left(\frac{1}{C_9}\right)^{\varepsilon(k+1)} \exp\left\{\varepsilon \frac{t_{k(n)}}{C_9(t_{k(n)} - [\bar{C}\ln n])}\right\} \left(1 + \zeta\left(\frac{C_5\bar{C}\ln n}{8M_\rho}\right)\right)^k \\
&\quad \times \left(\sum_{\ell=1}^{t_{k(n)}-[\bar{C}\ln n]-1} \exp\left\{\varepsilon \frac{t_{k(n)}}{C_9 \ell}\right\} \mathbb{P}\left(D^1(\ell)\right)\right)^k \\
&\leq \sum_{k=0}^{\infty} \left(\frac{1}{C_9}\right)^{\varepsilon(k+1)} e^{2\varepsilon/C_9} \left(1 + \zeta\left(\frac{C_5\bar{C}\ln n}{8M_\rho}\right)\right)^k \left(\sum_{\ell=1}^{b_n-1} \exp\left\{\varepsilon \frac{t_{k(n)}}{C_9\ell}\right\} \mathbb{P}\left(D^1(\ell)\right)\right)^k.
\end{aligned}$$

Since $b_n \sim t_{k(n)}$, we have $\sqrt{b_n} \sim t_{k(n)}^{1/2} \leq \bar{C}\ln n$. Similar to (3.63) and (3.64), we have, for $q < b_n$,

$$\begin{aligned}
&\sum_{\ell=1}^{q} \mathbb{P}(D^1(\ell)) \leq c_1 e^{-c_2 t_{k(n)}/q}, \\
&\sum_{\ell=1}^{b_n-1} \mathbb{P}(D^1(\ell)) \leq \mathbb{P}\left(\max_{1\leq k\leq b_n} \sum_{m=1}^{k} \rho_m \geq \frac{1}{8} C_5 \bar{C}\ln n\right) \leq c_3 < 1.
\end{aligned}$$

The same arguments as in (3.65) and (3.66) show that for large n such that $t_{k(n)} \leq 2(b_n - 1)$,

$$\begin{aligned}
&\sum_{\ell=1}^{b_n-1} \exp\left\{\varepsilon \frac{t_{k(n)}}{C_9\ell}\right\} \mathbb{P}\left(D^1(\ell)\right) \\
&\leq \sum_{\ell=1}^{b_n-2} \left(\exp\left\{\varepsilon \frac{t_{k(n)}}{C_9\ell}\right\} - \exp\left\{\varepsilon \frac{t_{k(n)}}{C_9(\ell+1)}\right\}\right) c_1 e^{-c_2 t_{k(n)}/\ell} + c_3 \exp\left\{\varepsilon \frac{t_{k(n)}}{C_9(b_n-1)}\right\} \\
&\leq c_1 \frac{\varepsilon}{C_9} \int_1^\infty e^{z(\varepsilon/C_9 - c_2)} \mathrm{d}z + c_3 \exp\left\{\varepsilon \frac{2}{C_9}\right\} =: F^1(\varepsilon).
\end{aligned}$$

Thus, there exists $n_0 > 0$ such that when $n \geq n_0$, it holds that

$$\mathbb{E}\left(\exp\left\{-\varepsilon 1_{\tilde{G}_n^1} \ln Y_n^1\right\}\right) \leq \frac{e^{2\varepsilon/C_9}}{(C_9)^\varepsilon} \sum_{k=0}^{\infty} \left(\frac{\left(1 + \zeta\left(C_5\bar{C}\ln n_0/(8M_\rho)\right)\right)}{(C_9)^\varepsilon} F^1(\varepsilon)\right)^k.$$

The rest of the proof is the same as the proof of (ii) in Lemma 3.12. □

Recall that $t_i = 2^i$, and L_i is defined in (3.38). For $n \in \mathbb{N}$, define

$$r_0^{(2,n)} := t_{i-1},$$

$$s_0^{(2,n)} := t_i \wedge \inf\left\{ k \geq r_0^{(2,n)} + 1 : \sum_{\ell = r_0^{(2,n)}+1}^{k} \overline{\rho}_{n-\ell}^{(n-k-[L_i/2]))} \geq \frac{C_5 t_i^{1/2}}{16} \right\}, \quad r_{j+1}^{(2,n)} := s_j^{(2,n)} + L_i,$$

$$s_{j+1}^{(2,n)} := \inf\left\{ k \geq r_{j+1}^{(2,n)} + 1 : \sum_{\ell = r_{j+1}^{(2,n)}+1}^{k} \overline{\rho}_{n-\ell}^{(n-k-[L_i/2])} \geq \frac{C_5 t_i^{1/2}}{16} \right\} \wedge (r_{j+1}^{(2,n)} + t_i - t_{i-1}),$$

$$\mathfrak{K}^{(2,n)} := \inf\{k : s_k^{(2,n)} \geq t_i\}.$$

Also define

$$G_i^{(2,n)} := \left\{ \max_{r \in [t_{i-1}, t_{i+1}], 0 < k \leq t_i^{1/2}/\alpha_1} \sum_{m=r+1}^{r+k} \overline{\rho}_{n-m}^{(n-t_{i+2})} \leq \frac{C_5 t_{i-1}^{1/2}}{32} \right\},$$

where α_1 is defined in (3.44).

When considering I_2, we need to be more careful since the definition of β_k^2 only makes sense for $k \geq 1$. Note that our definition of $s_j^{(2,n)}$ and $r_j^{(2,n)}$ does not need any restriction for n. Thus, the proof of (ii) for Lemma 3.14 below is the same as Lemma 3.12. Also note that our proof for (i) in Lemma 3.12 only relies on the property that the sequence of independent random variables $H_\ell^1 - H_{\ell-1}^1 - \widetilde{\Pi}_y^{\xi,-\lambda^*}\left[H_\ell^1 - H_{\ell-1}^1\right], \ell \geq 2$ is a sequence of centered sub-exponential random variables satisfying (3.31) and (3.33). The strong Markov property of Ξ^1 only used for the independence of H_ℓ^1. Note that for $k \geq 2$, we have

$$\beta_k^2 - \beta_{k-1}^2 = -\rho_k + H_k^2 - H_{k-1}^2 - \widetilde{\Pi}_y^{\xi,-\lambda^*}\left[H_k^2 - H_{k-1}^2\right] =: -\rho_k + \Delta H_k^2,$$

and for $|\eta| < \gamma(-\lambda^*) - (m-1)\mathrm{es}$,

$$\begin{aligned}
&\widetilde{\Pi}_y^{\xi,-\lambda^*}\left(e^{\eta \Delta H_k^2}\right) = \widetilde{\Pi}_{k-1}^{\xi,-\lambda^*}\left(\exp\left\{\eta\left(H_k^2 + \frac{1}{v^*}\left(\frac{\psi_\lambda(k,-\lambda^*,\omega)}{\psi(k,-\lambda^*,\omega)} - \frac{\psi_\lambda(k-1,-\lambda^*,\omega)}{\psi(k-1,-\lambda^*,\omega)}\right)\right)\right\}\right) \\
&= \widetilde{\Pi}_{k-1}^{\xi,-\lambda^*}\left(\exp\left\{\eta\left(H_k^2 + \frac{1}{v^*}\frac{\psi_\lambda(1,-\lambda^*,\theta_{k-1}\omega)}{\psi(1,-\lambda^*,\theta_{k-1}\omega)}\right)\right\}\right) \\
&= \exp\left\{\frac{\eta}{v^*}\frac{\psi_\lambda(1,-\lambda^*,\theta_{k-1}\omega)}{\psi(1,-\lambda^*,\theta_{k-1}\omega)}\right\}\Pi_0\left(\exp\left\{\int_0^{H_1}((m-1)\xi(B_s,\theta_{k-1}\omega) - \gamma(-\lambda^*) + \eta)\,\mathrm{d}s\right\}\right)
\end{aligned} \tag{3.69}$$

and the right-hand side of (3.69) also makes sense for $k \leq 1$. Therefore, to extend the case of β_k^2 for $k \leq 1$, we ignore the definition of β_0^2 and β_1^2 defined in (3.13), and we take a sequence of independent random variables $\left\{\Delta H_k^2 : k \leq 1\right\}$ independent to everything else with Laplace function given in (3.69) under $\widetilde{\Pi}_y^{\xi,-\lambda^*}$, then, or $k \leq 1$, define

$$\beta_k^2 := \beta_2^2 + \sum_{\ell=k+1}^{2}\left(\rho_\ell - \Delta H_\ell^2\right).$$

Replacing the definition of β_0^2 and β_1^2 defined in (3.13), by the above, Lemma 3.14(i) can be proven by an argument similar to that of Lemma 3.12(i).

Lemma 3.14. *(i) For sufficiently large i and $n \geq 2^{i-1}$, on $G_i^{(2,n)}$,*

$$\inf_{y \in [-1,1]} \inf_{x \geq \frac{C_5 t_{i-1}^{1/2}}{2}} \widetilde{\Pi}_y^{\xi,-\lambda^*}\Bigg(\left\{\beta_{n-t_i}^2 - \beta_n^2 \geq \frac{C_5 {t_i}^{1/2}}{2}\right\}$$

$$\bigcap \bigcap_{k=t_{i-1}+1}^{t_i} \{\beta_{n-k}^2 - \beta_n^2 \geq t_i^{1/4}\} \Big| \beta_{n-t_{i-1}}^2 - \beta_n^2 = x\Big)$$

$$\geq C_9^{\mathfrak{K}^{(2,n)}+1} \exp\left\{-\sum_{j=0}^{\mathfrak{K}^{(2,n)}} \frac{t_i}{C_9\left(s_j^{(2,n)} - r_j^{(2,n)}\right)}\right\} =: Z_i^{(2,n)}, \tag{3.70}$$

here C_9 is the constant in Lemma 3.12.

(ii) There exist constants $\varepsilon > 0$ and $K = K(\varepsilon)$ such that for n large enough,

$$\mathbb{E}\left(\exp\left\{-\varepsilon 1_{G_i^{(2,n)}} \ln Z_i^{(2,n)}\right\}\right) \leq K.$$

Proof. (i) For $r < s$ define

$$D^{(2,n)}(r,s) := \left\{\omega : \sum_{\ell=r+1}^{k} \overline{\rho}_{n-\ell}^{(n-k-[L_i/2])} < \frac{C_5 t_i^{1/2}}{16}, \forall k \in [r+1, s), \sum_{\ell=r+1}^{s} \overline{\rho}_{n-\ell}^{(n-s-[L_i/2])} \geq \frac{C_5 t_i^{1/2}}{16}\right\}.$$

Then by (3.37) we see that $D^{(2,n)}(r,s) \in \mathcal{F}_{n-r} \cap \mathcal{F}^{n-s-[L_i/2]}$. Define

$$D^{(2,n)}(\ell) := \left\{\omega : \sum_{\ell=1}^{k} \overline{\rho}_{n-\ell}^{(n-k-[L_i/2])} < \frac{C_5 t_i^{1/2}}{16}, \forall k \in [1, s), \sum_{\ell=1}^{s} \overline{\rho}_{n-\ell}^{(n-s-[L_i/2])} \geq \frac{C_5 t_i^{1/2}}{16}\right\}.$$

Similar to (3.41) and (3.68), we have

$$\begin{aligned}
&\mathbb{P}\left(s_j^{(2,n)} - r_j^{(2,n)} = \ell_j, 0 \leq j \leq k-1, s_k^{(2,n)} - r_k^{(2,n)} = \ell_k\right) \\
&\leq \mathbb{P}\left(s_j^{(2,n)} - r_j^{(2,n)} = \ell_j, 0 \leq j \leq k-1\right) \mathbb{P}\left(D^{(2,n)}(\ell_k)\right) \left(1 + \zeta\left(\frac{C_5 t_i^{1/2}}{32 M_\rho}\right)\right).
\end{aligned}$$

Define

$$\mathcal{A}_j^{(2,n)} := \left\{\beta_{n-s_j^{(2,n)}}^2 - \beta_n^2 \geq C_5 t_i^{1/2}\right\} \bigcap \bigcap_{k=r_j^{(2,n)}+1}^{s_j^{(2,n)}} \left\{\beta_{n-k}^2 - \beta_{n-r_j^{(2,n)}}^2 + \sum_{\ell=r_j^{(2,n)}+1}^{k} \rho_{n-\ell} \geq -\frac{C_5 t_i^{1/2}}{16}\right\},$$

$$\left(\mathcal{A}_j^{(2,n)}\right)' := \bigcap_{k=s_j^{(2,n)}+1}^{r_{j+1}^{(2,n)}} \left\{\beta_{n-k}^2 - \beta_{n-s_j^{(2,n)}}^2 + \sum_{\ell=s_j^{(2,n)}+1}^{k} \rho_{n-\ell} \geq -\frac{C_5 t_i^{1/2}}{16}\right\},$$

$$\mathcal{B}^{(2,n)} := \left(\bigcap_{j=0}^{\mathfrak{K}^{(2,n)}} \mathcal{A}_j^{(2,n)}\right) \bigcap \left(\bigcap_{j=0}^{\mathfrak{K}^{(2,n)}-1} \left(\mathcal{A}_j^{(2,n)}\right)'\right).$$

Then repeating the proof of Lemma 3.12 (i), we get (3.70).

(ii) Define

$$\mathfrak{J}^{(2,n)} := \inf\{j : s_j^{(2,n)} - r_j^{(2,n)} \geq t_i - t_{i-1}\}.$$

Although $s_j^{(2,n)} - r_j^{(2,n)}$ may depend on n, we see that the upper bound of (3.63) and (3.64) still hold if we replace $D(\ell_k)$ by $D^{(2,n)}(\ell_k)$ because of the stationarity of ρ. Repeating the argument in the proof of Lemma 3.12 (ii), we get that (ii) holds. The proof is now complete. □

Let L_n^1 be defined in Lemma 3.13. Define

$$r_0^{(2)} := [\bar{C} \ln n],$$

$$s_0^{(2)} := t_{k(n)} \wedge \inf\left\{k \geq r_0^{(2)} + 1 : \sum_{\ell=r_0^{(2)}+1}^{k} \overline{\rho}_{n-\ell}^{(n-k-[L_n^1/2])} \geq \frac{1}{4}C_5\bar{C}\ln n\right\}, \quad r_{j+1}^{(2)} := s_j^{(2)} + L_n^1$$

$$s_{j+1}^{(2)} := \inf\left\{k \geq r_{j+1}^{(2)} + 1 : \sum_{\ell=r_{j+1}^{(2)}+1}^{k} \overline{\rho}_{n-\ell}^{(n-k-[L_n^1/2])} \geq \frac{1}{4}C_5\bar{C}\ln n\right\} \wedge \left(r_{j+1}^{(2)} + t_{k(n)} - [\bar{C}\ln n]\right),$$

$$\mathfrak{K}^{(2)} := \inf\{k : s_k^{(2)} \geq t_{k(n)}\}.$$

Also define

$$\tilde{G}_n^2 := \left\{\max_{[\bar{C}\ln n] \leq r \leq 2t_{k(n)}, 0<k\leq[\bar{C}\ln n]/\alpha_2} \sum_{m=r+1}^{r+k} \overline{\rho}_{n-m}^{(n-4t_{k(n)})} \leq \frac{1}{8}C_5\bar{C}\ln n\right\},$$

here α_2 is defined in (3.44).

Lemma 3.15. *(i) For sufficiently large n, on $\tilde{G}_n^2$,*

$$\inf_{y\in[-1,1]} \inf_{x\geq C_5[\bar{C}\ln n]} \widetilde{\Pi}_y^{\xi,-\lambda^*}\Big\{\{\beta_{n-t_{k(n)}}^2 - \beta_n^2 \geq \frac{C_5 t_{k(n)}^{1/2}}{2}\}$$
$$\bigcap \bigcap_{k=[\bar{C}\ln n]+1}^{t_{k(n)}} \{\beta_{n-k}^2 - \beta_n^2 \geq 0\} \mid \beta_{n-[\bar{C}\ln n]}^2 - \beta_n^2 = x\Big\}$$
$$\geq C_9^{\mathfrak{K}^{(2)}+1} \exp\left\{-\sum_{j=0}^{\mathfrak{K}^{(2)}} \frac{t_{k(n)}}{C_9\left(s_j^{(2)} - r_j^{(2)}\right)}\right\} =: Y_n^2,$$

here C_9 is the constant in Lemma 3.12.

(ii) There exist positive constants ε and $K = K(\varepsilon)$ such that for n large enough,

$$\mathbb{E}\left(\exp\left\{-\varepsilon 1_{\tilde{G}_n^2} \ln Y_n^2\right\}\right) \leq K.$$

Proof. (i) Define

$$\mathcal{A}_j^{(2)} := \left\{\beta_{n-s_j^{(2)}}^2 - \beta_n^2 \geq 4C_5\bar{C}\ln n\right\} \bigcap \bigcap_{k=r_j^{(2)}+1}^{s_j^{(2)}} \left\{\beta_{n-k}^2 - \beta_{n-r_j^{(2)}}^2 + \sum_{\ell=r_j^{(2)}+1}^{k} \rho_{n-\ell} \geq -\frac{1}{4}C_5\bar{C}\ln n\right\},$$

$$\left(\mathcal{A}_j^{(2)}\right)' := \bigcap_{k=s_j^{(2)}+1}^{r_{j+1}^{(2)}} \left\{\beta_{n-k}^2 - \beta_{n-s_j^{(2)}}^2 + \sum_{\ell=s_j^{(2)}+1}^{k} \rho_{n-\ell} \geq -\frac{1}{4}C_5\bar{C}\ln n\right\},$$

$$\mathcal{B}^{(2)} := \left(\bigcap_{j=0}^{\mathfrak{K}^{(2)}} \mathcal{A}_j^{(2)}\right) \bigcap \left(\bigcap_{j=0}^{\mathfrak{K}^{(2)}-1} \left(\mathcal{A}_j^{(2)}\right)'\right).$$

Repeating the proof of in Lemma 3.12 (i), we get that (i) holds.

(ii) Define

$$\mathfrak{J}^{(2)} := \inf\{j : s_j^{(2)} - r_j^{(2)} \geq t_{k(n)} - [\bar{C}\ln n]\}.$$

Using an argument similar to that in the proof of Lemma 3.12 (ii), we get that (ii) holds. ☐

Proof of Lemma 3.3. Recall the definitions of I_1 and I_2 in (3.17). Recall that $k(n) = \left[\log_2\left([\bar{C}\ln n]^2\right)\right]$. Define $j(n) := \lceil\log_2 n\rceil$. Let $\bar{C}$ be a constant to be specified later.

By (3.36), there exists a large N_1 such that for $i, n \geq N_1$ and $\ell \geq t_{i+2} = 2^{i+2}$, we have that

$$\left\{ \max_{r\in[t_{i-1},t_{i+1}],0<k\leq t_i^{1/2}/\alpha_1} \sum_{m=r+1}^{r+k} \rho_m \leq \frac{C_5 t_{i-1}^{1/2}}{64} \right\} \subset G_i^1,$$

$$\left\{ \max_{[\bar{C}\ln n]\leq r\leq 2[\bar{C}\ln n]^2,0<k\leq[\bar{C}\ln n]/\alpha_2} \sum_{m=r+1}^{r+k} \rho_m \leq \frac{1}{16} C_5 \bar{C} \ln n \right\} \subset \widetilde{G}_n^1,$$

$$\left\{ \max_{r\in[t_{i-1},t_{i+1}],0<k\leq t_i^{1/2}/\alpha_1} \sum_{m=r+1}^{r+k} \rho_{\ell-m} \leq \frac{C_5 t_{i-1}^{1/2}}{64} \right\} \subset G_i^{(2,\ell)},$$

$$\left\{ \max_{[\bar{C}\ln n]\leq r\leq 2[\bar{C}\ln n]^2,0<k\leq[\bar{C}\ln n]/\alpha_2} \sum_{m=r+1}^{r+k} \rho_{n-m} \leq \frac{1}{16} C_5 \bar{C} \ln n \right\} \subset \widetilde{G}_n^2.$$

Take an integer N_2 such that $N_2 \geq 2^{N_1+2}$. By Lemma 3.8, (3.62) and the stationarity of ρ_m, we have

$$\begin{aligned}
&\sum_{n=N_2}^{\infty} \sum_{i=k(n)}^{j(n)} \mathbb{P}\left((G_i^1)^c\right) + \sum_{\ell=N_2}^{\infty} \sum_{i=k(\ell)}^{j(\ell)} \mathbb{P}\left((G_i^{(2,\ell)})^c\right) \\
&\leq 2 \sum_{n=N_2}^{\infty} \sum_{i=k(n)}^{j(n)} t_{i+1} \mathbb{P}\left(\max_{0\leq k\leq t_i^{1/2}/\alpha_1} \sum_{m=1}^{k} \rho_m > \frac{C_5 t_{i-1}^{1/2}}{64} \right) \\
&\leq 2 \sum_{n=N_2}^{\infty} 2n \sum_{i=k(n)}^{j(n)} c_1 \exp\left\{ -\frac{\alpha_1 C_5^2 t_{i-1}}{64^2 c_1 t_i^{1/2}} \right\} \\
&\leq 4c_1 \sum_{n=N_2}^{\infty} n \log_2(n) \exp\left\{ -\frac{\alpha_1 C_5^2 \bar{C} \ln n}{64^2 c_1} \right\} < \infty,
\end{aligned} \tag{3.71}$$

and

$$\begin{aligned}
&\sum_{n=N_1}^{\infty} \mathbb{P}\left(\left(\tilde{G}_n^1\right)^c\right) + \sum_{n=N_1}^{\infty} \mathbb{P}\left(\left(\tilde{G}_n^2\right)^c\right) \\
&\leq 2 \sum_{n=N_1}^{\infty} 2[\bar{C}\ln n]^2 \mathbb{P}\left(\max_{0\leq k\leq[\bar{C}\ln n]/\alpha_2} \sum_{m=1}^{k} \rho_m > \frac{1}{16} C_5 \bar{C} \ln n \right) \\
&\leq 4 \sum_{n=N_1}^{\infty} [\bar{C}\ln n]^2 \cdot c_1 \exp\left\{ -\frac{\alpha_2 C_5^2 [\bar{C}\ln n]}{16^2 c_1} \right\} < \infty,
\end{aligned} \tag{3.72}$$

by taking $\bar{C}$ large enough.

Recall the definition of Z_i^1 and G_i^1 in Lemma 3.12. By (3.40), (3.37), we have that for all $i \in \mathbb{Z}_+$ large enough such that $t_{i-1} > L_i$, $t_i > t_i^{1/2}/\alpha_1$ and $t_{i-1} - [L_i/2] > t_{i-2}$,

$$\begin{aligned}
G_i^1 &\in \mathcal{F}^{t_{i-2}} \cap \mathcal{F}_{t_{i+1}+t_i^{1/2}/\alpha_1} \subset \mathcal{F}^{t_{i-2}} \cap \mathcal{F}_{t_{i+1}+t_i}, \\
Z_i^1 &\in \mathcal{F}^{t_{i-1}-[L_i/2]} \cap \mathcal{F}_{t_{i+1}} \subset \mathcal{F}^{t_{i-2}} \cap \mathcal{F}_{t_{i+1}+t_i}.
\end{aligned}$$

Then when i is large enough, $Z_{4i}^1, G_{4i}^1 \in \mathcal{F}_{t_{4i+1}+t_{4i}} \cap \mathcal{F}^{t_{4i-2}}$. Together with Lemma 3.12, for large i, we have

$$\mathbb{E}\left(\exp\left\{-\varepsilon 1_{G_{4i}^1} \ln Z_{4i}^1\right\} \middle| \mathcal{F}_{t_{4i-3}+t_{4i-4}}\right) \leq \left(1+\zeta(2^{4i-4})\right) \mathbb{E}\left(\exp\left\{-\varepsilon 1_{G_{4i}^1} \ln Z_{4i}^1\right\}\right) \leq K\left(1+\zeta(0)\right).$$

Thus, we have, for $k = 0, 1, 2, 3$,

$$\mathbb{E}\left(\exp\left\{-\varepsilon \sum_{i=\lceil k(n)/4\rceil}^{[j(n)/4]-1} 1_{G^1_{4i}} \ln Z_{4i+k}\right\}\right) \le \prod_{i=\lceil k(n)/4\rceil}^{[j(n)/4]-1} K\left(1+\zeta(0)\right) \le c_1^{j(n)}$$

for some constant $c_1 > 0$. Now for a constant $c_2 > 0$, we have

$$\begin{aligned}\mathbb{P}\left(\sum_{i=k(n)+1}^{j(n)} 1_{G^1_i} \ln Z^1_i < -c_2 j(n)\right) &\le \sum_{k=0}^{3} \mathbb{P}\left(\sum_{i=\lceil k(n)/4\rceil}^{[j(n)/4]-1} 1_{G^1_{4i+k}} \ln Z_{4i+k} < -\frac{c_2 j(n)}{4}\right)\\ &\le 4e^{-\varepsilon c_2 j(n)/4} \max_{k=1,2,3,4} \mathbb{E}\left(\exp\left\{-\varepsilon \sum_{i=\lceil k(n)/4\rceil}^{[j(n)/4]-1} 1_{G^1_{4i+k}} \ln Z_{4i+k}\right\}\right) \le 4e^{-\varepsilon c_2 j(n)/4}\cdot c_1^{j(n)}.\end{aligned}$$

Letting c_2 be large enough so that $-\varepsilon c_2/4 + \ln c_1 < -\ln 2$, there exists N_3 such that

$$\sum_{n=N_3}^{\infty} \mathbb{P}\left(\sum_{i=k(n)+1}^{j(n)} 1_{G^1_i} \ln Z^1_i < -c_2 j(n)\right) \le \sum_{n=N_3}^{\infty} 4e^{-\varepsilon c_2 j(n)/4}\cdot c_1^{j(n)} < \infty. \tag{3.73}$$

Similarly, by Lemma 3.13, Lemma 3.14 and Lemma 3.15, we have

$$\begin{aligned}&\sum_{n=N_3}^{\infty} \mathbb{P}\left(1_{\tilde{G}^1_n} \ln Y^1_n < -c_2 \ln n\right) + \sum_{n=N_3}^{\infty} \mathbb{P}\left(\sum_{i=k(n)+1}^{j(n)} 1_{G^{(2,n)}_i} \ln Z^{(2,n)}_i < -c_2 j(n)\right)\\ &\quad + \sum_{n=N_3}^{\infty} \mathbb{P}\left(1_{\tilde{G}^2_n} \ln Y^2_n < -c_2 \ln n\right) < \infty.\end{aligned} \tag{3.74}$$

Now define

$$\begin{aligned}\widetilde{\Omega} :=&\Omega_2 \bigcap \left\{(G^1_n)^c \text{ i.o.}\right\}^c \bigcap \left\{(\widetilde{G}^1_n)^c \text{ i.o.}\right\}^c \bigcap \left\{\left(\bigcap_{i=k(n)+1}^{j(n)} G^{(2,n)}_i\right)^c \text{ i.o.}\right\}^c \bigcap \left\{(\widetilde{G}^2_n)^c \text{ i.o.}\right\}^c\\ &\bigcap \left\{\left\{\sum_{i=k(n)+1}^{j(n)} 1_{G^1_i} \ln Z^1_i < -c_2 j(n)\right\} \text{ i.o.}\right\}^c \bigcap \left\{\left\{1_{\widetilde{G}^1_n} \ln Y^1_n < -c_2 \ln n\right\} \text{ i.o.}\right\}^c\\ &\bigcap \left\{\left\{\sum_{i=k(n)+1}^{j(n)} 1_{G^{(2,n)}_i} \ln Z^{(2,n)}_i < -c_2 j(n)\right\} \text{ i.o.}\right\}^c \bigcap \left\{\left\{1_{\widetilde{G}^2_n} \ln Y^2_n < -c_2 \ln n\right\} \text{ i.o.}\right\}^c.\end{aligned}$$

Then by (3.71), (3.72), (3.73) and (3.74), $\mathbb{P}(\widetilde{\Omega}) = 1$. By the construction of $\widetilde{\Omega}$, for every $\omega \in \widetilde{\Omega}$, there exists $N = N(\omega)$ such that when $n \ge N$, we have

$$1_{G^1_n} = 1_{\tilde{G}^1_n} = \prod_{i=k(n)+1}^{j(n)} 1_{G^{(2,n)}_i} = 1_{\tilde{G}^2_n} = 1,$$

and

$$\begin{aligned}\sum_{i=k(n)+1}^{j(n)} 1_{G^1_i} \ln Z^1_i \ge -c_2 j(n), &\quad 1_{\widetilde{G}^1_n} \ln Y^1_n \ge -c_2 \ln n,\\ \sum_{i=k(n)+1}^{j(n)} 1_{G^{(2,n)}_i} \ln Z^{(2,n)}_i \ge -c_2 j(n), &\quad 1_{\widetilde{G}^2_n} \ln Y^2_n \ge -c_2 \ln n.\end{aligned}$$

Now for $\omega \in \widetilde{\Omega}$ and $n \geq N$, it holds that uniformly for $y \in [-1,1]$,

$$\begin{aligned}
&\widetilde{\Pi}_y^{\xi,-\lambda^*}\left(\beta_k^1-\beta_1^1 \geq 0,\ \forall 1 \leq k \leq n, \beta_n^1-\beta_1^1 \geq n^{1/4}\right)\\
&\geq \prod_{k=1}^{[\bar{C}\ln n]} \widetilde{\Pi}_y^{\xi,-\lambda^*}\left(\beta_k^1-\beta_{k-1}^1 > C_5\right)\cdot Y_n^1 \cdot \prod_{i=k(n)+1}^{j(n)} Z_i^1\\
&\geq C_5^{[\bar{C}\ln n]}\cdot \exp\left\{1_{\widetilde{G}_n^1}\ln Y_n^1\right\}\cdot \exp\left\{\sum_{i=k(n)+1}^{j(n)} 1_{G_i^1}\ln Z_i^1\right\}\\
&\geq C_5^{[\bar{C}\ln n]}\cdot e^{-c_2\ln n}\cdot e^{-c_2 j(n)} \geq \frac{1}{C_5} n^{\bar{C}\ln C_5-c_2-c_2/\ln 2},
\end{aligned}$$

where in the second inequality we used Lemma 3.8. Taking $[V_t]=n$, and $\ell'' > -\bar{C}\ln C_5 + c_2 + c_2/\ln 2$, we get the first result.

The second result can be proved similarly. Note that $n \geq 2^{j(n)-1} = t_{j(n)-1}$ holds for every $n > 2$. When $n \geq N(\omega)$, by Lemma 3.14, uniformly for $y \in [-1,1]$,

$$\begin{aligned}
&\widetilde{\Pi}_y^{\xi,-\lambda^*}\left(\beta_{n-k}^2-\beta_n^2 \geq 0,\ \forall 1 \leq k \leq n, \beta_2^2-\beta_n^2 \geq n^{1/4}\right)\\
&\geq \prod_{k=1}^{[\bar{C}\ln n]} \widetilde{\Pi}_y^{\xi,-\lambda^*}\left(\beta_{n-k}^2-\beta_{n-k+1}^2 > C_5\right)\cdot Y_n^2 \cdot \prod_{i=k(n)+1}^{j(n)} Z_i^{(2,n)}\\
&\geq C_5^{[\bar{C}\ln n]}\cdot \exp\left\{1_{\widetilde{G}_n^2}\ln Y_n^2\right\}\cdot \exp\left\{\sum_{i=k(n)+1}^{j(n)} 1_{G_i^{(2,n)}}\ln Z_i^{(2,n)}\right\}\\
&\geq \frac{1}{C_5} n^{\bar{C}\ln C_5-c_2-c_2/\ln 2}.
\end{aligned}$$

Taking $[V_t]=n$ and $\ell'' > -\bar{C}\ln C_5 + c_2 + c_2/\ln 2$ completes the proof. □

4 Appendix

Lemma 4.1. *For* $j=1,2$, *let* β_k^j *be define by* (3.13). *Under* $\widetilde{\Pi}_y^{\xi,-\lambda^*}$, $\{\beta_k^j, k \geq 1\}$ *has independent increments.*

Proof. We only need to prove the result for $j=1$. For any $n \geq 2$, $1 \leq k_0 < k_1 < \ldots < k_n$, we only need to prove that $H_{k_1}^1 - H_{k_0}^1, \ldots, H_{k_n}^1 - H_{k_{n-1}}^1$ are independent. For any bounded measurable functions $f_1, \ldots, f_n$, by the strong Markov property of $\left\{\Xi_t, t \geq 0; \widetilde{\Pi}_y^{\xi,-\lambda^*}\right\}$, we have

$$\begin{aligned}
&\widetilde{\Pi}_y^{\xi,-\lambda^*}\left(\prod_{j=1}^n f_j\left(H_{k_j}^1-H_{k_{j-1}}^1\right)\middle|\mathcal{F}_{H_{k_0}^1}\right) = \widetilde{\Pi}_{k_0}^{\xi,-\lambda^*}\left(f_1\left(H_{k_1}^1\right)\prod_{j=2}^n f_j\left(H_{k_j}^1-H_{k_{j-1}}^1\right)\right)\\
&= \widetilde{\Pi}_{k_0}^{\xi,-\lambda^*}\left(f_1\left(H_{k_1}^1\right)\widetilde{\Pi}_{k_0}^{\xi,-\lambda^*}\left(\prod_{j=2}^n f_j\left(H_{k_j}^1-H_{k_{j-1}}^1\right)\middle|\mathcal{F}_{H_{k_1}^1}\right)\right)\\
&= \widetilde{\Pi}_{k_0}^{\xi,-\lambda^*}\left(f_1\left(H_{k_1}^1\right)\widetilde{\Pi}_{k_1}^{\xi,-\lambda^*}\left(f_2\left(H_{k_2}^1\right)\prod_{j=3}^n f_j\left(H_{k_j}^1-H_{k_{j-1}}^1\right)\right)\right)\\
&= \widetilde{\Pi}_{k_0}^{\xi,-\lambda^*}\left(f_1\left(H_{k_1}^1\right)\right)\widetilde{\Pi}_{k_1}^{\xi,-\lambda^*}\left(f_2\left(H_{k_2}^1\right)\prod_{j=3}^n f_j\left(H_{k_j}^1-H_{k_{j-1}}^1\right)\right)
\end{aligned}$$

$$
\begin{aligned}
&= \widetilde{\Pi}_{k_0}^{\xi,-\lambda^*}\left(f_1\left(H^1_{k_1}\right)\right)\widetilde{\Pi}_{k_1}^{\xi,-\lambda^*}\left(f_2\left(H^1_{k_2}\right)\widetilde{\Pi}_{k_1}^{\xi,-\lambda^*}\left(\prod_{j=3}^n f_j\left(H^1_{k_j}-H^1_{k_{j-1}}\right)\Big|\mathcal{F}_{H^1_{k_2}}\right)\right)\\
&= \widetilde{\Pi}_{k_0}^{\xi,-\lambda^*}\left(f_1\left(H^1_{k_1}\right)\right)\widetilde{\Pi}_{k_1}^{\xi,-\lambda^*}\left(f_2\left(H^1_{k_2}\right)\right)\widetilde{\Pi}_{k_2}^{\xi,-\lambda^*}\left(f_3\left(H^1_{k_3}\right)\prod_{j=4}^n f_j\left(H^1_{k_j}-H^1_{k_{j-1}}\right)\right)\\
&= \cdots = \prod_{j=1}^n \widetilde{\Pi}_{k_{j-1}}^{\xi,-\lambda^*}\left(f_j\left(H^1_{k_j}\right)\right).
\end{aligned}
$$

Taking expectation in the display above, we get

$$
\widetilde{\Pi}_{y}^{\xi,-\lambda^*}\left(\prod_{j=1}^n f_j\left(H^1_{k_j}-H^1_{k_{j-1}}\right)\right) = \prod_{j=1}^n \widetilde{\Pi}_{k_{j-1}}^{\xi,-\lambda^*}\left(f_j\left(H^1_{k_j}\right)\right). \tag{4.1}
$$

In particular, taking $f_1 = ... = f_{n-1} = 1$, we get

$$
\widetilde{\Pi}_{y}^{\xi,-\lambda^*}\left(f_n\left(H^1_{k_n}-H^1_{k_{n-1}}\right)\right) = \widetilde{\Pi}_{k_{n-1}}^{\xi,-\lambda^*}\left(f_n\left(H^1_{k_n}\right)\right).
$$

Thus, (4.1) can be rewritten as

$$
\widetilde{\Pi}_{y}^{\xi,-\lambda^*}\left(\prod_{j=1}^n f_j\left(H^1_{k_j}-H^1_{k_{j-1}}\right)\right) = \prod_{j=1}^n \widetilde{\Pi}_{y}^{\xi,-\lambda^*}\left(f_j\left(H^1_{k_j}-H^1_{k_{j-1}}\right)\right), \tag{4.2}
$$

which says that $\left(H^1_{k_1}-H^1_{k_0}\right), ..., \left(H^1_{k_n}-H^1_{k_{n-1}}\right)$ are independent. □

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