



Local properties for 1-dimensional critical branching Lévy process[☆]

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ABSTRACT

Consider a one dimensional critical branching Lévy process $((Z_t)_{t \geq 0}, \mathbb{P}_x)$. Assume that the offspring distribution either has finite second moment or belongs to the domain of attraction of some α -stable distribution with $\alpha \in (1, 2)$, and that the underlying Lévy process $(\xi_t)_{t \geq 0}$ is non-lattice and has finite $2 + \delta^*$ moment for some $\delta^* > 0$. We first prove that

$$t^{\frac{1}{\alpha-1}} \left(1 - \mathbb{E}_{\sqrt{t}y} \left(\exp \left\{ -\frac{1}{t^{\frac{1}{\alpha-1}-\frac{1}{2}}} \int h(x) Z_t(dx) - \frac{1}{t^{\frac{1}{\alpha-1}}} \int g \left(\frac{x}{\sqrt{t}} \right) Z_t(dx) \right\} \right) \right)$$

converges as $t \rightarrow \infty$ for any non-negative bounded Lipschitz function g and any non-negative directly Riemann integrable function h of compact support. Then for any $y \in \mathbb{R}$ and bounded Borel set A of positive Lebesgue measure with its boundary having zero Lebesgue measure, under a higher moment condition on ξ , we find the decay rate of the probability $\mathbb{P}_{\sqrt{t}y}^{\cdot}(Z_t(A) > 0)$. As an application, we prove some convergence results for Z_t under the conditional law $\mathbb{P}_{\sqrt{t}y}^{\cdot}(\cdot | Z_t(A) > 0)$.

1. Introduction and main results

1.1. Background introduction and motivation

A branching random walk is a discrete-time Markov process which can be described as follows. At time 0 there is a particle at $x \in \mathbb{R}^d$. At time 1, this particle dies and gives birth to N offspring with $\mathbb{P}(N = k) = p_k$ for $k \in \mathbb{N} := \{0, 1, \dots\}$, and the relative positions of the offspring to the parent are given by iid copies of a random variable \mathbf{X} . These offspring form generation 1. Given the information at time 1, at time 2, individuals of generation 1 independently repeat their parent's behavior. The procedure goes on. Let Z_n be the counting measure of the individuals of generation n . $(Z_n)_{n \geq 0}$ is called a branching random walk starting from an initial individual located at x . We will use \mathbb{P}_x to denote the law of the branching random walk and \mathbb{E}_x to denote the corresponding expectation.

Let $C^+(\mathbb{R})$ be the family of non-negative continuous functions on \mathbb{R} and $C_c^+(\mathbb{R})$ be the subfamily of functions in $C^+(\mathbb{R})$ with compact support. For any $f \in C_c^+(\mathbb{R})$, we write $\ell(f) := \int f(x) dx$. Assume that the branching random walk is critical, that is,

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$\sum_{k=0}^{\infty} k p_k = 1$ and $p_1 < 1$. It is well-known that this process will become extinct with probability 1. For any $\mathbf{x} \in \mathbb{R}^d$, we use $\|\mathbf{x}\|$ to denote the Euclidean norm. When $d \geq 3$, under the assumption

$$\sum_{k=0}^{\infty} k^2 p_k < \infty \tag{1.1}$$

and that either \mathbf{X} is a standard \mathbb{R}^d -valued Gaussian random variable or \mathbf{X} is a bounded symmetric \mathbb{Z}^d -valued random variable, Rapenne [1, Lemma 2.10] proved that for any closed ball $\mathbf{A} \subset \mathbb{R}^d$, there exists a constant $I_{\mathbf{A}}$ such that for all $\mathbf{x} \in \mathbb{Z}^d$, as $n \rightarrow \infty$,

$$\lim_{n \rightarrow \infty} n^{d/2} \mathbb{P}_{[\sqrt{n}]\mathbf{x}}(Z_n(\mathbf{A}) > 0) = \frac{I_{\mathbf{A}}}{\sqrt{\det(\Sigma)}} \exp\left\{-\frac{1}{2} \mathbf{x}^T \Sigma^{-1} \mathbf{x}\right\}, \tag{1.2}$$

where $\Sigma = (\Sigma_{i,j})_{1 \leq i,j \leq d}$ is the covariance matrix of \mathbf{X} , i.e., $\Sigma_{i,j} = \text{Cov}(X_i, X_j)$ for all $1 \leq i, j \leq d$. Moreover, under the same assumption, Rapenne [1, Proposition 2.13] proved that for any $\mathbf{a} \in \mathbb{Z}^d$ and closed ball $\mathbf{A} \subset \mathbb{R}^d$, there exists a random point process $(\mathcal{N}_{\mathbf{A}}, \mathbb{P})$ supported in \mathbf{A} and independent of \mathbf{a} such that

$$\mathbb{P}_{[\sqrt{n]}\mathbf{a}}(Z_n \in \cdot | Z_n(\mathbf{A}) > 0) \xrightarrow{d} \mathbb{P}(\mathcal{N}_{\mathbf{A}} \in \cdot). \tag{1.3}$$

For critical branching Brownian motions and critical super-Brownian motions in dimension $d \geq 3$, results similar to (1.2) and (1.3) are consequences of [2, (2.8)]. More precisely, taking $f = \theta 1_{\mathbf{A}}$ in [2, (2.8)] and then letting $\theta \rightarrow \infty$, we get (1.2); taking a general $f \in C_c^+(\mathbb{R})$ and combining it with (1.2), we get (1.3).

When $d = 2$, things are quite different. When (1.1) holds and \mathbf{X} is a \mathbb{Z}^2 -valued random variable such that $\mathbb{P}(\|\mathbf{X}\| \leq 1) = 1$, Lalley and Zheng [3, Propositions 31 and 33] proved that for any $\mathbf{x} \in \mathbb{Z}^2$, there exists $C(\mathbf{x}) > 0$ such that for all $n \geq 2$,

$$\frac{1}{C(\mathbf{x})} \leq n(\log n) \mathbb{P}_{[\sqrt{n}]\mathbf{x}}(Z_n(\{\mathbf{0}\}) > 0) \leq C(\mathbf{x}). \tag{1.4}$$

Recently Chen et al. [4] refined the result of (1.4). They proved that if $\sum_{k=0}^{\infty} e^{\varepsilon k} p_k < \infty$ for some $\varepsilon > 0$, then for any $\mathbf{x} \in \mathbb{Z}^2$,

$$\lim_{n \rightarrow \infty} n(\log n) \mathbb{P}_{[\sqrt{n}]\mathbf{x}}(Z_n(\{\mathbf{0}\}) > 0) = \frac{4}{\sigma^2} e^{-\frac{\varepsilon}{4} \|\mathbf{x}\|^2}, \tag{1.5}$$

where $\sigma^2 := \sum_{k=0}^{\infty} k^2 p_k - 1$. Comparing (1.2) and (1.5), we see that there is an extra factor $\log n$ in $d = 2$. In the case of critical continuous-time binary branching random walk $(Z_t)_{t \geq 0}$ with branching rate 2, under a second moment condition on the random walk, Durrett [5, (8.12)] proved that for any bounded open set $\mathbf{A} \subset \mathbb{R}^2$ with $\ell(\partial \mathbf{A}) = 0$ and any $\theta > 0$,

$$\lim_{t \rightarrow \infty} t(\log t) \left(1 - \mathbb{E} \left(\exp \left\{ -\frac{\theta}{\log t} \frac{Z_t(\mathbf{A})}{\ell(\mathbf{A})} \right\} \right) \right) = \frac{4\theta}{\theta + 8\pi}, \tag{1.6}$$

where ℓ is the Lebesgue measure. As a consequence of (1.6), see [5, (8.15)], for any bounded open set $\mathbf{A} \subset \mathbb{R}^2$ with $\ell(\partial \mathbf{A}) = 0$ and any $h > 0$, it holds that

$$\lim_{t \rightarrow \infty} t(\log t) \mathbb{P} \left(\frac{1}{\log t} \frac{Z_t(\mathbf{A})}{\ell(\mathbf{A})} > h \right) = 4e^{-8\pi h}. \tag{1.7}$$

In the case $d = 1$, it is well-known (for example, see the paragraph below [6, Theorem 3], or [7]) that, if the assumption (1.1) holds and $\mathbb{E}(X) = 0, \mathbb{E}(X^2) = 1$, there exists a measure-valued random variable (Y, \mathbb{P}) such that as $n \rightarrow \infty$,

$$\mathbb{P}(Z_1^{(n)} \in \cdot | Z_n(\mathbb{R}) > 0) \xrightarrow{d} \mathbb{P}(Y \in \cdot), \tag{1.8}$$

where $Z_1^{(n)}$ is the random measure defined by $\int f(x) Z_1^{(n)}(dx) := \frac{1}{n} \int f\left(\frac{x}{\sqrt{n}}\right) Z_n(dx)$ for all bounded non-negative functions f . The random measure Y is related to a certain super-Brownian motion, which will be introduced later. It is easy to see from (1.8) that for any bounded non-negative continuous function f on \mathbb{R} ,

$$\begin{aligned} & \lim_{n \rightarrow \infty} \frac{1}{\mathbb{P}(Z_n(\mathbb{R}) > 0)} \left(1 - \mathbb{E} \left(\exp \left\{ -\frac{1}{n} \int f\left(\frac{x}{\sqrt{n}}\right) Z_n(dx) \right\} \right) \right) \\ &= 1 - \mathbb{E} \left(\exp \left\{ -\int f(x) Y(dx) \right\} \right). \end{aligned} \tag{1.9}$$

We will prove a result more general than (1.9) in Theorem 1.2 below in the continuous time setting. To the best of our knowledge, there are no $d = 1$ counterparts yet to the high dimensional results (1.2) and (1.3). In this paper, we will prove the counterparts of (1.2) and (1.3) for 1-dimensional critical branching Lévy processes, see Theorems 1.4 and 1.6 below.

A branching Lévy process is a continuous counterpart of branching random walk and it can be described as follows. At time 0, there is an individual at $\mathbf{x} \in \mathbb{R}^d$ and it moves according to a Lévy process $(\xi_t, \mathbf{P}_{\mathbf{x}})$. After an exponential time with parameter $\beta > 0$, this individual dies and gives birth to k offspring with probability $p_k, k = 0, 1, \dots$ located at the parent's death place. The offspring then independently repeat the parent's behavior. This procedure goes on. Let Z_t be the point process formed by the individual alive at time t . The process $(Z_t)_{t \geq 0}$ is called a branching Lévy process.

A more general class of branching Lévy processes on the real line has been introduced by Bertoin and Mallein [8]. Recently, quite a few papers, see, for instance, [9–12], have been devoted to this general model. In this general model the branching rate

β is allowed to be infinite and that the branching is allowed to be non-local, i.e., the birthplace of the children is not necessarily the deathplace of their parent. The branching Lévy process dealt with in this paper is a special case of the branching Lévy process introduced in [8] with local branching and finite branching rate.

We will use \mathbb{P}_x to denote the law of $(Z_t)_{t \geq 0}$ starting from one individual at x and \mathbb{E}_x to denote the corresponding expectation. For simplicity, we write $\mathbb{P} := \mathbb{P}_0$ and $\mathbb{E} := \mathbb{E}_0$. We will assume that $(Z_t)_{t \geq 0}$ is critical:

$$\sum_{k=0}^{\infty} k p_k = 1 \text{ and } p_1 < 1.$$

The main purpose of this paper is to study the asymptotic behavior of $(Z_t)_{t \geq 0}$ when $d = 1$ under some conditions. We will assume that

(H1) The offspring distribution $\{p_k : k \geq 0\}$ belongs to the domain of attraction of an α -stable, $\alpha \in (1, 2]$, distribution. More precisely, either there exist $\alpha \in (1, 2)$ and $\kappa(\alpha) \in (0, \infty)$ such that

$$\lim_{n \rightarrow \infty} n^\alpha \sum_{k=n}^{\infty} p_k = \kappa(\alpha),$$

or that (corresponding $\alpha = 2$)

$$\sum_{k=0}^{\infty} k^2 p_k < \infty.$$

Under the assumption **(H1)**, it is known (see, for example, [13–15]) that

$$\lim_{t \rightarrow \infty} t^{\frac{1}{\alpha-1}} \mathbb{P}(Z_t(\mathbb{R}) > 0) = \begin{cases} \frac{\alpha-1}{\beta \kappa(\alpha) \Gamma(2-\alpha)}, & \text{when } \alpha \in (1, 2), \\ \frac{2}{\beta (\sum_{k=1}^{\infty} k(k-1)p_k)}, & \alpha = 2, \end{cases} \tag{1.10}$$

For the Lévy process $(\xi_t)_{t \geq 0}$, we will assume that

(H2)

$$\mathbf{E}_0(\xi_1) = 0, \quad \mathbf{E}_0(\xi_1^2) = 1;$$

(H3) the law of ξ_1 under \mathbf{P}_0 is non-lattice;

and that

(H4) there exists $\delta^* > 0$ such that $\mathbf{E}_0(|\xi_1|^{2+\delta^*}) < \infty$.

The hypothesis **(H3)** and **(H4)** will only be used to prove Lemma 2.2 below. For some results, we will also need the following stronger moment condition on the Lévy process:

(H4') For the $\alpha \in (1, 2]$ in **(H1)**, it holds that

$$\mathbf{E}_0(|\xi_1|^{r_0}) < \infty \quad \text{for some } r_0 > \frac{2\alpha}{\alpha-1}.$$

When $\alpha = 2$, the assumption is the same as that in [6]. For any $t > 0$, define M_t to be the maximal position of all the particles at time t . We also define

$$M := \sup_{t>0} M_t.$$

Under **(H1)**, **(H2)** and the weaker moment condition $\mathbf{E}_0((\xi_1 \vee 0)^{r_0}) < \infty$ than **(H4')**, it was proved in [16] (although [16] did not deal with the case $\alpha = 2$, the proof is actually the same as the case $\alpha \in (1, 2)$, see the argument below [16, Theorem 1.1]) that there exists $\theta(\alpha) \in (0, \infty)$ such that

$$\lim_{x \rightarrow \infty} x^{\frac{2}{\alpha-1}} \mathbb{P}(M \geq x) = \theta(\alpha). \tag{1.11}$$

The assumption **(H4')** is only used in the proof of Lemma 3.5 to control the overshoot of the underlying Lévy process.

1.2. Critical super-Brownian motion

In this subsection, we give a brief introduction to super-Brownian motion. Let $\mathcal{M}_F(\mathbb{R})$ be the families of finite Borel measures on \mathbb{R} . We will use $\mathbf{0}$ to denote the null measure on \mathbb{R} . Let $B_b^+(\mathbb{R})$ be the space of non-negative bounded Borel functions on \mathbb{R} . For any $f \in B_b^+(\mathbb{R})$ and $\mu \in \mathcal{M}_F(\mathbb{R})$, we use $\mu(f)$ to denote the integral of f with respect to μ . For any $\alpha \in (1, 2]$, the function

$$\varphi(\lambda) := C(\alpha)\lambda^\alpha := \begin{cases} \frac{\beta \kappa(\alpha) \Gamma(2-\alpha)}{\alpha-1} \lambda^\alpha, & \text{when } \alpha \in (1, 2), \\ \frac{\beta}{2} (\sum_{k=1}^{\infty} k(k-1)p_k) \lambda^2, & \alpha = 2, \end{cases} \tag{1.12}$$

where $\kappa(\alpha)$ is the constant given in (H1) and $\Gamma(z) := \int_0^\infty t^{z-1} e^{-t} dt$ is the Gamma function, is a branching mechanism. Since $\varphi'(0) = 0$, φ is a critical branching mechanism. Let (B_t, \mathbf{P}_x) be a standard Brownian motion.

The critical super-Brownian motion $X = \{(X_t)_{t \geq 0}; \mathbb{P}_\mu\}$ that we will use in this paper is an $\mathcal{M}_F(\mathbb{R})$ -valued Markov process such that for any $f \in B_b^+(\mathbb{R})$,

$$-\log \mathbb{E}_\mu \left(\exp \{ -X_t(f) \} \right) = \mu \left(v_f^X(t, \cdot) \right),$$

where $(t, x) \mapsto v_f^X(t, x)$ is the unique locally bounded non-negative solution to

$$v_f^X(t, x) = \mathbf{E}_x \left(f(B_t) \right) - \mathbf{E}_y \left(\int_0^t \varphi \left(v_f^X(t-s, B_s) \right) ds \right). \tag{1.13}$$

Since $1 < \frac{2}{\alpha-1}$, by [17, Theorem 1.2] and [18], for any $\mu \in \mathcal{M}_F(\mathbb{R})$, \mathbb{P}_μ -almost surely, the random measure X_t is absolutely continuous with respect to the Lebesgue measure and the density function

$$Y_t(x) := \frac{X_t(dx)}{dx}$$

has a version which is continuous in x . We will always use Y_t to denote this version.

For the probabilistic representation of the weak convergence limit via super Brownian motion in Theorem 1.6 below, we will also need the \mathbb{N} -measures of the super Brownian motion X .

Without loss of generality, we assume that X is the coordinate process on

$$\mathbb{D} := \{w = (w_t)_{t \geq 0} : w \text{ is an } \mathcal{M}_F(\mathbb{R})\text{-valued càdlàg function}\}.$$

We assume that $(\mathcal{F}_\infty, (\mathcal{F}_t)_{t \geq 0})$ is the natural filtration on \mathbb{D} , completed as usual with the \mathcal{F}_∞ -measurable and \mathbb{P}_μ -negligible sets for all $\mu \in \mathcal{M}_F(\mathbb{R})$. Let \mathbb{W}_0^+ be the family of $\mathcal{M}_F(\mathbb{R})$ -valued càdlàg functions on $(0, \infty)$ with $\mathbf{0}$ as a trap and with $\lim_{t \downarrow 0} w_t = \mathbf{0}$.

Since the super Brownian motion X is critical and that $\int_1^\infty \frac{1}{\varphi(\lambda)} d\lambda < \infty$, we see that $\mathbb{P}_{\delta_y}(X_t = \mathbf{0}) > 0$ for all $t > 0$ and $y \in \mathbb{R}$, which implies that there exists a unique family of σ -finite measures $\{\mathbb{N}_y; y \in \mathbb{R}\}$ on \mathbb{W}_0^+ such that for any $\mu \in \mathcal{M}_F(\mathbb{R})$, if $\mathcal{N}(dw)$ is a Poisson random measure on \mathbb{W}_0^+ with intensity measure

$$\mathbb{N}_\mu(dw) := \int_{\mathbb{R}} \mathbb{N}_y(dw) \mu(dy),$$

then the process defined by

$$\hat{X}_0 := \mu, \quad \hat{X}_t := \int_{\mathbb{W}_0^+} w_t \mathcal{N}(dw), \quad t > 0,$$

is a realization of the superprocess $X = \{(X_t)_{t \geq 0}; \mathbb{P}_\mu\}$. Furthermore, for any $t > 0$, $y \in \mathbb{R}$ and $f \in B_b^+(\mathbb{R})$,

$$\mathbb{N}_y \left(1 - \exp \{ -w_t(f) \} \right) = -\log \mathbb{E}_{\delta_y} \left(\exp \{ -X_t(f) \} \right), \tag{1.14}$$

see [19] or [20, Theorems 8.27 and 8.28]. The next useful result says that for any given $t > 0$ and $y \in \mathbb{R}$, w_t has an \mathbb{N}_y -a.e. continuous density.

Define

$$\mathcal{A} := \left\{ \mu \in \mathcal{M}_F(\mathbb{R}) : \frac{d\mu}{dx} \in C^+(\mathbb{R}) \right\}.$$

Lemma 1.1. *For any $t > 0$ and $y \in \mathbb{R}$, it holds that*

$$\mathbb{N}_y(w_t \notin \mathcal{A}) = 0.$$

The proof is postponed to Section 4. We still use $\{Y_t(x), x \in \mathbb{R}\}$ to denote the density of w_t .

1.3. Main results

We will sometimes use $\ell(A)$ to denote the Lebesgue measure of a Borel set $A \subset \mathbb{R}$. Let $\text{DRI}^+(\mathbb{R})$ ($\text{DRI}_c^+(\mathbb{R})$) be the family of non-negative directly Riemann integrable functions (of compact support). We say that a bounded Borel set A is directly Riemann integrable if the indicator 1_A is a directly Riemann integrable function. It is well known that (i) any directly Riemann integrable function is bounded; (ii) a non-negative Borel function of compact support is directly Riemann integrable if and only if it is Riemann integrable and (iii) a bounded Borel set A is directly Riemann integrable if and only if $\ell(\partial A) = 0$. For the definition of directly Riemann integrable function, see the beginning of Section 2.2. Let $B_{Lip}^+(\mathbb{R})$ be the family of bounded non-negative Lipschitz continuous functions in \mathbb{R} . For any $g \in B_{Lip}^+(\mathbb{R})$, we use $\text{Lip}(g)$ to denote its Lipschitz constant.

Theorem 1.2. *If (H1), (H2), (H3) and (H4) hold, then for any $y \in \mathbb{R}$, $g \in B_{Lip}^+(\mathbb{R})$ and $h \in \text{DRI}_c^+(\mathbb{R})$, it holds that*

$$\lim_{t \rightarrow \infty} t^{\frac{1}{\alpha-1}} \left(1 - \mathbb{E}_{\sqrt{t}y} \left(\exp \left\{ -\frac{1}{t^{\frac{1}{\alpha-1}-\frac{1}{2}}} \int h(x) Z_t(dx) - \frac{1}{t^{\frac{1}{\alpha-1}}} \int g \left(\frac{x}{\sqrt{t}} \right) Z_t(dx) \right\} \right) \right)$$

$$= -\log \mathbb{E}_{\delta_y} \left(\exp \left\{ -\ell(h)Y_1(0) - X_1(g) \right\} \right).$$

Remark 1.3. In the special case $\alpha = 2$, taking $h = 0$ in [Theorem 1.2](#), we get that

$$\begin{aligned} & \lim_{t \rightarrow \infty} t \left(1 - \mathbb{E}_{\sqrt{t}y} \left(\exp \left\{ -\frac{1}{t} \int g \left(\frac{x}{\sqrt{t}} \right) Z_t(dx) \right\} \right) \right) \\ &= -\log \mathbb{E}_{\delta_y} \left(\exp \left\{ -X_1(g) \right\} \right) = \mathbb{N}_y \left(1 - \exp \left\{ -w_1(g) \right\} \right), \end{aligned} \tag{1.15}$$

where in the last equality we used [\(1.14\)](#). Combining [\(1.10\)](#) and [\(1.15\)](#), we get that

$$\begin{aligned} & \lim_{t \rightarrow \infty} \frac{1}{\mathbb{P}(Z_t(\mathbb{R}) > 0)} \left(1 - \mathbb{E}_{\sqrt{t}y} \left(\exp \left\{ -\frac{1}{t} \int g \left(\frac{x}{\sqrt{t}} \right) Z_t(dx) \right\} \right) \right) \\ &= C(2)\mathbb{N}_y \left(1 - \exp \left\{ -w_1(g) \right\} \right). \end{aligned} \tag{1.16}$$

It follows from [\(1.14\)](#) and [\[21, \(1.11\)\]](#) that $\mathbb{N}_y(w(1) > 0) = C(2)^{-1}$. Therefore, by [\(1.16\)](#), we conclude that

$$\begin{aligned} & \lim_{t \rightarrow \infty} \frac{1}{\mathbb{P}(Z_t(\mathbb{R}) > 0)} \left(1 - \mathbb{E}_{\sqrt{t}y} \left(\exp \left\{ -\frac{1}{t} \int g \left(\frac{x}{\sqrt{t}} \right) Z_t(dx) \right\} \right) \right) \\ &= \mathbb{N}_y \left(1 - \exp \left\{ -w_1(g) \right\} \mid w(1) > 0 \right). \end{aligned} \tag{1.17}$$

Combining [\(1.9\)](#) and [\(1.17\)](#), we immediately get that $(Y, \mathbb{P}) \stackrel{d}{=} (w_1, \mathbb{N}_0(\cdot \mid w_1(1) > 0))$.

In the special case $\alpha = 2$, taking $g = 0$ and $h(x) = \frac{\theta 1_A(x)}{\ell(A)}$ with $\theta > 0$, A being a bounded Borel set with $\ell(A) > 0$ and $\ell(\partial A) = 0$ in [Theorem 1.2](#), we get

$$\lim_{t \rightarrow \infty} t \left(1 - \mathbb{E} \left(\exp \left\{ -\frac{\theta}{\sqrt{t}} \frac{Z_t(A)}{\ell(A)} \right\} \right) \right) = -\log \mathbb{E}_{\delta_y} \left(\exp \left\{ -\theta Y_1(0) \right\} \right).$$

Comparing the result above with [\(1.6\)](#) and [\(1.7\)](#) for $d = 2$, we see the differences between the cases $d = 2$ and $d = 1$. In the case $d = 2$, there is an extra factor $\log t$ in the decay, and also one needs to normalize with $\log t$ instead of \sqrt{t} . In addition, the limit in $d = 2$ is related to the Laplace transform of an exponential random variable while in the case $d = 1$, the limit is related to super-Brownian motion.

Theorem 1.4. If [\(H1\)](#), [\(H2\)](#), [\(H3\)](#) and [\(H4'\)](#) hold, then for any $y \in \mathbb{R}$ and any bounded Borel set A with $\ell(A) > 0$ and $\ell(\partial A) = 0$, it holds that

$$\lim_{t \rightarrow \infty} t^{\frac{1}{\alpha-1}} \mathbb{P}_{\sqrt{t}y} \left(Z_t(A) > 0 \right) = -\log \mathbb{P}_{\delta_y} \left(Y_1(0) = 0 \right).$$

Remark 1.5. (i) Taking $h = \theta 1_A$ and $g = 0$ in [Theorem 1.2](#) first and then letting $\theta \uparrow \infty$, formally one expects the limit in [Theorem 1.4](#) to be $-\log \mathbb{P}_{\delta_y} \left(\ell(A)Y_1(0) = 0 \right)$. Under the assumption $\ell(A) > 0$, the limit is equal to $-\log \mathbb{P}_{\delta_y} \left(Y_1(0) = 0 \right)$, which implies that the right-hand side of the limit in [Theorem 1.4](#) is independent of A .

(ii) When $\alpha = 2$, [Theorem 1.4](#) is the 1-d counterpart to the high dimensional result [\(1.2\)](#) and [\(1.5\)](#). We see that branching plays a more important role in dimension 1 while spatial motion dominates in dimension $d \geq 2$. In dimension 1, the limit is related to the density of super-Brownian motion, while in dimension $d \geq 3$, the limit in [\(1.2\)](#) is only related to the local limit of a random walk (see [\[1, Proposition 2.1\]](#)) and that branching only appears in the constant I_A (see the end of the proof of [\[1, Lemma 2.10\]](#) on page 14). In dimension 2, both the branching and the spatial motion affect the limit in [\(1.5\)](#) in the sense that the limit requires at least second moment due to the appearance of σ^2 and that the exponential term $e^{-\frac{5}{4}|x|^2}$ is related to the local limit theorem for the random walk.

For any $t > 0$, we define a measure $Z_1^{(t)}$ by

$$\int f(y)Z_1^{(t)}(dy) := \frac{1}{t^{\frac{1}{\alpha-1}}} \int f \left(\frac{y}{\sqrt{t}} \right) Z_t(dy).$$

The next result is an application of [Theorems 1.2](#) and [1.4](#).

Theorem 1.6. Assume that [\(H1\)](#), [\(H2\)](#), [\(H3\)](#) and [\(H4'\)](#) hold. Suppose that $y \in \mathbb{R}$ and A is a bounded Borel set with $\ell(A) > 0$ and $\ell(\partial A) = 0$.

(i) As $t \rightarrow \infty$, we have

$$\left(\frac{1}{t^{\frac{1}{\alpha-1}-\frac{1}{2}}} Z_t, \mathbb{P}_{\sqrt{t}y}(\cdot \mid Z_t(A) > 0) \right) \xrightarrow{d} (Y_1(0)\ell, \mathbb{N}_y(\cdot \mid Y_1(0) > 0))$$

in the sense of vague topology.

(ii) As $t \rightarrow \infty$, it holds that

$$\left(Z_t^{(t)}, \mathbb{P}_{\sqrt{t}y}(\cdot | Z_t(A) > 0) \right) \xrightarrow{d} (w_1, \mathbb{N}_y(\cdot | Y_1(0) > 0))$$

in the sense of weak topology.

Remark 1.7. In the special case $\alpha = 2$, Theorem 1.6 (i) is the 1-d counterpart to the high dimensional result (1.3). There are some differences between the 1-d case and the high dimensional case. First there is an extra factor \sqrt{t} in the 1-d case while no normalization is needed in the high dimensional case $d \geq 3$. Also in the 1-d case, the limit is an absolutely continuous random measure (with respect to the Lebesgue measure) with density $Y_1(0)$ (the density $Y_1(x)$ of super-Brownian motion X_1 evaluated at 0), while in the high dimensional case $d \geq 3$, the limit \mathcal{N}_A is a random point measure supported on A .

Theorem 1.6 (ii) should be compared with (1.8). (1.8) is about the asymptotic of Z_t conditioned on global survival $Z_t(\mathbb{R}) > 0$, while Theorem 1.6 (ii) is about the asymptotic of Z_t conditioned on local $Z_t(A) > 0$. As we mentioned in Remark 1.3, in the special case $\alpha = 2$, the limit (Y, \mathbb{P}) in (1.8) is equal in law to $(w_1, \mathbb{N}_0(\cdot | w_1(1) > 0))$ which is different from the limit $(w_1, \mathbb{N}_y(\cdot | Y_1(0) > 0))$ in Theorem 1.6 (ii).

Theorem 1.6 (i) describes the local behavior of the counting measure Z_t , while Theorem 1.6 (ii) is about the global behavior of Z_t .

After our paper was accepted, we found that Lee [22] has already studied local properties of critical binary branching Brownian motion in all dimensions $d \geq 1$. In the present paper, our spatial motion is a general Lévy process and our branching mechanism is only assumed to be in domain of attraction of a stable distribution. Moreover, our proof strategies are different from [22].

We end this section with a brief description of the organization of this paper. In Section 2, we give some elementary estimates involving the standard normal density and about the underlying Lévy process. We also derive an integral equation for the Laplace transform of Z_t and prove the existence and uniqueness of solution for the problem (2.25) below. In Section 3, we give the proofs of Theorems 1.2, 1.4 and 1.6. In Section 4, we give the proof of Lemma 1.1.

For two functions $f(x)$ and $g(x)$ with $x \in E$, we use $f \lesssim g, x \in E$, to denote that there exists a constant C independent of x such that $f(x) \leq Cg(x), x \in E$.

2. Preliminaries

2.1. Some estimates involving the standard normal density

Throughout this paper, $\phi(x) := \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$ is the standard normal density.

Lemma 2.1.

(i) For any $\Delta > 0$,

$$\sup_{y \in \mathbb{R}} |\phi(y) - \phi(y + \Delta)| \leq (\Delta \wedge \sqrt{\Delta}).$$

(ii) For any $0 < r < s$ with $s - r \in (0, 1)$, it holds that

$$\sup_{y \in \mathbb{R}} \left| \frac{1}{\sqrt{r}} \phi\left(\frac{y}{\sqrt{r}}\right) - \frac{1}{\sqrt{s}} \phi\left(\frac{y}{\sqrt{s}}\right) \right| \leq \frac{1}{\sqrt{r}} - \frac{1}{\sqrt{s}} + \frac{1}{\sqrt{s}} \left(\sqrt{s-r} + 1 - \exp\left\{-\frac{\sqrt{s-r}}{r}\right\} \right).$$

Proof. (i) It is easy to check that

$$|\phi'(x)| = \frac{1}{\sqrt{2\pi}} |x| e^{-\frac{x^2}{2}} \leq 1.$$

Therefore, noticing that $\phi(x) \leq \frac{1}{\sqrt{2\pi}} \leq \frac{1}{2}$, we conclude that

$$\sup_{y \in \mathbb{R}} |\phi(y) - \phi(y + \Delta)| \leq \Delta \wedge 1 \leq (\Delta \wedge \sqrt{\Delta}).$$

(ii) For any $y \in \mathbb{R}$,

$$\begin{aligned} & \left| \frac{1}{\sqrt{r}} \phi\left(\frac{y}{\sqrt{r}}\right) - \frac{1}{\sqrt{s}} \phi\left(\frac{y}{\sqrt{s}}\right) \right| \leq \left| \frac{1}{\sqrt{r}} - \frac{1}{\sqrt{s}} \right| \phi\left(\frac{y}{\sqrt{r}}\right) + \frac{1}{\sqrt{s}} \left| \phi\left(\frac{y}{\sqrt{r}}\right) - \phi\left(\frac{y}{\sqrt{s}}\right) \right| \\ & \leq \frac{1}{\sqrt{2\pi}} \left| \frac{1}{\sqrt{r}} - \frac{1}{\sqrt{s}} \right| + \frac{1}{\sqrt{2\pi s}} \exp\left\{-\frac{y^2}{2s}\right\} \left(1 - \exp\left\{-y^2 \left(\frac{1}{2r} - \frac{1}{2s}\right)\right\} \right) \\ & \leq \frac{1}{\sqrt{r}} - \frac{1}{\sqrt{s}} + \frac{1}{\sqrt{s}} \exp\left\{-\frac{y^2}{2s}\right\} \left(1 - \exp\left\{-y^2 \left(\frac{1}{2r} - \frac{1}{2s}\right)\right\} \right). \end{aligned} \tag{2.1}$$

If $y^2\sqrt{s-r} > 2s$, then by the inequality $ae^{-a} < 1$ for all $a > 1$, we get that

$$\exp\left\{-\frac{y^2}{2s}\right\} \left(1 - \exp\left\{-y^2\left(\frac{1}{2r} - \frac{1}{2s}\right)\right\}\right) \leq \exp\left\{-\frac{1}{\sqrt{s-r}}\right\} \leq \sqrt{s-r}. \tag{2.2}$$

If $y^2\sqrt{s-r} \leq 2s$, then

$$\begin{aligned} \exp\left\{-\frac{y^2}{2s}\right\} \left(1 - \exp\left\{-y^2\left(\frac{1}{2r} - \frac{1}{2s}\right)\right\}\right) &\leq 1 - \exp\left\{-\frac{s}{\sqrt{s-r}} \cdot \frac{s-r}{sr}\right\} \\ &= 1 - \exp\left\{-\frac{\sqrt{s-r}}{r}\right\}. \end{aligned} \tag{2.3}$$

Combining (2.1), (2.2) and (2.3), we conclude that

$$\left|\frac{1}{\sqrt{r}}\phi\left(\frac{y}{\sqrt{r}}\right) - \frac{1}{\sqrt{s}}\phi\left(\frac{y}{\sqrt{s}}\right)\right| \leq \frac{1}{\sqrt{r}} - \frac{1}{\sqrt{s}} + \frac{1}{\sqrt{s}} \left(\sqrt{s-r} + 1 - \exp\left\{-\frac{\sqrt{s-r}}{r}\right\}\right),$$

which implies the desired result. \square

The following inequality will be used several times later:

$$\left|1 - e^{-(x+y)} - x - y\right| \leq x^2 + y^2, \quad x, y \geq 0. \tag{2.4}$$

The proof of this inequality is elementary and we omit it.

2.2. Estimates for the Lévy process

We first give a local limit theorem for the underlying Lévy process $(\xi_t)_{t \geq 0}$. Before that, we recall the definition of directly Riemann integrable functions. For more details on properties of directly Riemann integrable functions, one can refer to [23, Section XI.1] and [24, Section 2.1].

Let f be a non-negative Borel function. For any $\kappa > 0$, define

$$\begin{aligned} \bar{f}_\kappa(x) &:= \sum_{m \in \mathbb{Z}} 1_{[m\kappa, (m+1)\kappa)}(x) \sup_{z \in [m\kappa, (m+1)\kappa)} f(z), \\ \underline{f}_\kappa(x) &:= \sum_{m \in \mathbb{Z}} 1_{[m\kappa, (m+1)\kappa)}(x) \inf_{z \in [m\kappa, (m+1)\kappa)} f(z). \end{aligned}$$

We say that f is directly Riemann integrable if $\int \bar{f}_\kappa(x) dx < \infty$ for some $\kappa > 0$ and

$$\lim_{\kappa \rightarrow 0} \int_{\mathbb{R}} (\bar{f}_\kappa(x) - \underline{f}_\kappa(x)) dx = 0.$$

Recall that we use $\text{DRI}^+(\mathbb{R})$ to denote the family of non-negative directly Riemann integrable functions. It is easy to see from the definition that any $h \in \text{DRI}^+(\mathbb{R})$ must be bounded. For $h \in \text{DRI}^+(\mathbb{R})$, we define $\|h\|_\infty := \sup_{x \in \mathbb{R}} |h(x)|$.

Lemma 2.2. Assume that (H2), (H3) and (H4) hold. For any $f \in \text{DRI}^+(\mathbb{R})$, it holds that

$$\lim_{n \rightarrow \infty} \sup_{x \in \mathbb{R}} \left| \sqrt{n} \mathbf{E}_x(f(\xi_n)) - \ell(f)\phi\left(\frac{x}{\sqrt{n}}\right) \right| = 0,$$

where $\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$ is the standard normal density.

Proof. For any $\kappa, \vartheta > 0$, define

$$\bar{f}_{\kappa, \vartheta}(x) := \sup_{|y| \leq \vartheta} \bar{f}_\kappa(x+y) \quad \text{and} \quad \underline{f}_{\kappa, \vartheta}(x) := \inf_{|y| \leq \vartheta} \underline{f}_\kappa(x+y),$$

then by [24, Lemma 2.2], it holds that

$$\lim_{\kappa \rightarrow 0} \lim_{\vartheta \rightarrow 0} \int |\bar{f}_{\kappa, \vartheta}(x) - f(x)| dx = \lim_{\kappa \rightarrow 0} \lim_{\vartheta \rightarrow 0} \int |\underline{f}_{\kappa, \vartheta}(x) - f(x)| dx = 0. \tag{2.5}$$

Let $\vartheta \in (0, \frac{1}{2})$ be sufficiently small, then by [24, (2.6) and Theorem 2.7], there exist a constant $K > 0$ independent of ϑ and κ and a constant $C_\vartheta > 0$ independent of κ such that for any $\kappa > 0, x \in \mathbb{R}$,

$$\mathbf{E}_0(f(x + \xi_n)) - \frac{1 + K\vartheta}{\sqrt{n}} \int \bar{f}_{\kappa, \vartheta}(x+z)\phi\left(\frac{z}{\sqrt{n}}\right) dz \leq \frac{C_\vartheta}{n^{(1+\delta^*)/2}} \int \bar{f}_{\kappa, \vartheta}(x+z) dz \tag{2.6}$$

and that

$$\begin{aligned} & \mathbf{E}_0(f(x + \xi_n)) - \frac{1}{\sqrt{n}} \int \left(\underline{f}_{\kappa, \vartheta}(x + z) - K\vartheta f(x + z) \right) \phi\left(\frac{z}{\sqrt{n}}\right) dz \\ & \geq -\frac{C_\vartheta}{n^{(1+\delta^*)/2}} \int f(x + z) dz. \end{aligned} \tag{2.7}$$

Therefore, by (2.6), we see that

$$\begin{aligned} & \sqrt{n} \mathbf{E}_x(f(\xi_n)) - \int f(x + z) \phi\left(\frac{z}{\sqrt{n}}\right) dz \\ & \leq \left(K\vartheta + \frac{C_\vartheta}{n^{\delta^*/2}} \right) \int \bar{f}_{\kappa, \vartheta}(x + z) dz + \int \left| \bar{f}_{\kappa, \vartheta}(x) - f(x) \right| dx \\ & \leq \left(K\vartheta + \frac{C_\vartheta}{n^{\delta^*/2}} + 1 \right) \int \left| \bar{f}_{\kappa, \vartheta}(x) - f(x) \right| dx + \left(K\vartheta + \frac{C_\vartheta}{n^{\delta^*/2}} \right) \int f(x) dx =: \bar{I}(\vartheta, \kappa, n). \end{aligned} \tag{2.8}$$

Similarly, according to (2.7), we have

$$\begin{aligned} & \sqrt{n} \mathbf{E}_x(f(\xi_n)) - \int f(x + z) \phi\left(\frac{z}{\sqrt{n}}\right) dz \\ & \geq -\frac{C_\vartheta}{n^{\delta^*/2}} \int f(x + z) dz - K\vartheta \int f(x + z) \phi\left(\frac{z}{\sqrt{n}}\right) dz - \int \left| \underline{f}_{\kappa, \vartheta}(x) - f(x) \right| dx \\ & \geq -\left(K\vartheta + \frac{C_\vartheta}{n^{\delta^*/2}} \right) \int f(x) dx - \int \left| \underline{f}_{\kappa, \vartheta}(x) - f(x) \right| dx \geq -\underline{I}(\vartheta, \kappa, n). \end{aligned} \tag{2.9}$$

Therefore, combining (2.8) and (2.9), we conclude that

$$\begin{aligned} & \limsup_{n \rightarrow \infty} \sup_{x \in \mathbb{R}} \left| \sqrt{n} \mathbf{E}_x(f(\xi_n)) - \int f(x + z) \phi\left(\frac{z}{\sqrt{n}}\right) dz \right| \\ & \leq \lim_{n \rightarrow \infty} \bar{I}(\vartheta, \kappa, n) + \lim_{n \rightarrow \infty} \underline{I}(\vartheta, \kappa, n) \\ & = (K\vartheta + 1) \int \left| \bar{f}_{\kappa, \vartheta}(x) - f(x) \right| dx + 2K\vartheta \int f(x) dx + \int \left| \underline{f}_{\kappa, \vartheta}(x) - f(x) \right| dx. \end{aligned}$$

By (2.5), letting $\vartheta \rightarrow 0$ first and then $\kappa \rightarrow 0$ in the above inequality, we get

$$\lim_{n \rightarrow \infty} \sup_{x \in \mathbb{R}} \left| \sqrt{n} \mathbf{E}_x(f(\xi_n)) - \int f(x + z) \phi\left(\frac{z}{\sqrt{n}}\right) dz \right| = 0.$$

Thus, to prove the desired result, it remains to show that

$$\lim_{n \rightarrow \infty} \sup_{x \in \mathbb{R}} \left| \phi\left(\frac{x}{\sqrt{n}}\right) \int f(z) dz - \int f(x + z) \phi\left(\frac{z}{\sqrt{n}}\right) dz \right| = 0.$$

Let E be any bounded interval such that $\text{supp}(f) \subset E$, then for any $x \in \mathbb{R}$,

$$\begin{aligned} & \left| \phi\left(\frac{x}{\sqrt{n}}\right) \int f(z) dz - \int f(x + z) \phi\left(\frac{z}{\sqrt{n}}\right) dz \right| \\ & = \left| \phi\left(\frac{x}{\sqrt{n}}\right) \int f(z) dz - \int f(z) \phi\left(\frac{x - z}{\sqrt{n}}\right) dz \right| \\ & \leq \|f\|_\infty \int_E \left| \phi\left(\frac{x}{\sqrt{n}}\right) - \phi\left(\frac{x - z}{\sqrt{n}}\right) \right| dz \leq \ell(E) \|f\|_\infty \sup_{z \in E} \left| \phi\left(\frac{x}{\sqrt{n}}\right) - \phi\left(\frac{x - z}{\sqrt{n}}\right) \right|. \end{aligned}$$

If $|x| < n^{2/3}$, then by the inequality $e^{-a} - e^{-b} \leq |b - a|$ for any $a, b \geq 0$, we see that

$$\begin{aligned} & \sup_{z \in E} \left| \phi\left(\frac{x}{\sqrt{n}}\right) - \phi\left(\frac{x - z}{\sqrt{n}}\right) \right| \leq \frac{1}{\sqrt{2\pi}} \sup_{z \in E} \left| \frac{x^2 - (x - z)^2}{n} \right| \\ & \leq \frac{2}{\sqrt{2\pi}} \left(\frac{1}{n} \sup_{z \in E} z^2 + \frac{1}{n^{1/3}} \sup_{z \in E} |z| \right) \xrightarrow{n \rightarrow \infty} 0. \end{aligned}$$

On the other hand, if $|x| > n^{2/3}$, then for large n , we see that

$$\sup_{z \in E} \left| \phi\left(\frac{x}{\sqrt{n}}\right) - \phi\left(\frac{x - z}{\sqrt{n}}\right) \right| \leq \phi(n^{1/6}) + \phi\left(\frac{n^{2/3} - \sup_{z \in E} |z|}{\sqrt{n}}\right) \xrightarrow{n \rightarrow \infty} 0.$$

The proof is now complete. \square

Remark 2.3. We mention here that the non-lattice assumption **(H3)** is only used to prove Lemma 2.2. If **(H3)** does not hold, it is possible to get a result similar to Lemma 2.2. For example, if ξ is a compound Poisson process supported on \mathbb{Z} with $\mathbf{E}_0(\xi_1) = 0, \mathbf{E}_0(\xi_1^2) = 1, \mathbf{E}_0(|\xi_1|^3) < \infty$ and that the support of the Lévy measure contains $\{n, n + 1\}$ for some $n \in \mathbb{N}$, then by [25, Theorem 13, p.206], for any $a \in \mathbb{Z}$, it is easily seen that

$$\limsup_{n \rightarrow \infty} \sup_{x \in \mathbb{Z}} \left| \sqrt{n} \mathbf{P}_x(\xi_n = a) - \phi\left(\frac{x}{\sqrt{n}}\right) \right| = 0.$$

Replace the Lebesgue measure ℓ by the counting measure ℓ_c on \mathbb{Z} , and for any bounded function f with compact support, define

$$\ell_c(f) := \sum_{i \in \mathbb{Z}} f(i).$$

Denote by $B_c^+(\mathbb{Z})$ the class of non-negative bounded functions with compact support. In this case, we see that for any $f \in B_c^+(\mathbb{Z})$,

$$\limsup_{n \rightarrow \infty} \sup_{x \in \mathbb{Z}} \left| \sqrt{n} \mathbf{E}_x(f(\xi_n)) - \ell_c(f) \phi\left(\frac{x}{\sqrt{n}}\right) \right| = 0.$$

Then the conclusions of Theorems 1.2, 1.4 and 1.6 remain true if $h \in \text{DRI}_c^+(\mathbb{R})$ is replaced by $h \in B_c^+(\mathbb{Z})$, ℓ replaced by ℓ_c and A replaced by $B \subset \mathbb{Z}$.

Lemma 2.4. Assume that **(H2)**, **(H3)** and **(H4)** hold. For any $h \in \text{DRI}_c^+(\mathbb{R})$, it holds that

$$\limsup_{t \rightarrow \infty} \sup_{x \in \mathbb{R}} \left| \sqrt{t} \mathbf{E}_x(h(\xi_t)) - \ell(h) \phi\left(\frac{x}{\sqrt{t}}\right) \right| = 0.$$

Proof. We write $t > 1$ as $t = [t] + \gamma$ with $\gamma \in [0, 1)$. Combining the Markov property and the inequality $\sqrt{t} - \sqrt{[t]} = \frac{t - [t]}{\sqrt{t} + \sqrt{[t]}} \leq \frac{1}{\sqrt{t}}$, we see that for any $x \in \mathbb{R}$,

$$\begin{aligned} & \left| \sqrt{t} \mathbf{E}_x(h(\xi_t)) - \ell(h) \phi\left(\frac{x}{\sqrt{t}}\right) \right| \\ & \leq (\sqrt{t} - \sqrt{[t]}) \|h\|_\infty + \mathbf{E}_x \left(\left| \sqrt{[t]} \mathbf{E}_{\xi_\gamma}(h(\xi_{[t]})) - \ell(h) \phi\left(\frac{\xi_\gamma}{\sqrt{[t]}}\right) \right| \right) + \ell(h) \left| \mathbf{E}_x \left(\phi\left(\frac{\xi_\gamma}{\sqrt{[t]}}\right) \right) - \phi\left(\frac{x}{\sqrt{t}}\right) \right| \\ & \leq \frac{\|h\|_\infty}{\sqrt{t}} + \sup_{z \in \mathbb{R}} \left| \sqrt{[t]} \mathbf{E}_z(h(\xi_{[t]})) - \ell(h) \phi\left(\frac{z}{\sqrt{[t]}}\right) \right| + \ell(h) \sup_{z \in \mathbb{R}} \left| \mathbf{E}_z \left(\phi\left(\frac{\xi_\gamma}{\sqrt{[t]}}\right) \right) - \phi\left(\frac{z}{\sqrt{t}}\right) \right|. \end{aligned}$$

The first term on the right-hand side tends to 0 as $t \rightarrow \infty$. By Lemma 2.2, the second term also tends to 0 as $t \rightarrow \infty$. Therefore, it remains to prove that

$$\limsup_{t \rightarrow \infty} \sup_{x \in \mathbb{R}} \left| \mathbf{E}_x \left(\phi\left(\frac{\xi_\gamma}{\sqrt{[t]}}\right) \right) - \phi\left(\frac{x}{\sqrt{t}}\right) \right| = \limsup_{t \rightarrow \infty} \sup_{x \in \mathbb{R}} \left| \mathbf{E}_0 \left(\phi\left(\frac{\xi_\gamma + x}{\sqrt{[t]}}\right) \right) - \phi\left(\frac{x}{\sqrt{t}}\right) \right| = 0. \tag{2.10}$$

Note that for $|x| > t^{2/3}$,

$$\begin{aligned} & \left| \mathbf{E}_0 \left(\phi\left(\frac{\xi_\gamma + x}{\sqrt{[t]}}\right) \right) - \phi\left(\frac{x}{\sqrt{t}}\right) \right| \leq \mathbf{E}_0 \left(\phi\left(\frac{\xi_\gamma + x}{\sqrt{[t]}}\right) \right) + \phi\left(\frac{x}{\sqrt{t}}\right) \\ & \leq \mathbf{P}_0 \left(\sup_{s \leq 1} |\xi_s| > \sqrt{t} \right) + \phi\left(\frac{t^{2/3} - \sqrt{t}}{\sqrt{[t]}}\right) + \phi(t^{1/6}) \xrightarrow{t \rightarrow \infty} 0. \end{aligned} \tag{2.11}$$

For $|x| \leq t^{2/3}$, by the inequality $|e^{-x^2} - e^{-y^2}| \leq |x^2 - y^2|$, we have

$$\begin{aligned} & \left| \mathbf{E}_0 \left(\phi\left(\frac{\xi_\gamma + x}{\sqrt{[t]}}\right) \right) - \phi\left(\frac{x}{\sqrt{t}}\right) \right| \leq \frac{1}{\sqrt{2\pi}} \mathbf{E}_0 \left(\left| \frac{x^2 - (\xi_\gamma + x)^2}{2[t]} \right| \right) + \frac{x^2}{\sqrt{2\pi}} \left(\frac{1}{2[t]} - \frac{1}{2t} \right) \\ & \leq \frac{1}{2[t]\sqrt{2\pi}} \mathbf{E}_0(\xi_\gamma^2 + 2|\xi_\gamma||x|) + \frac{t^{1/3}}{2\sqrt{2\pi}[t]} \leq \frac{1}{2[t]\sqrt{2\pi}} (\gamma + 2\sqrt{\gamma}t^{2/3}) + \frac{t^{1/3}}{2\sqrt{2\pi}[t]} \\ & \leq \frac{1}{2[t]\sqrt{2\pi}} (1 + 2t^{2/3}) + \frac{t^{1/3}}{2\sqrt{2\pi}[t]} \xrightarrow{t \rightarrow \infty} 0. \end{aligned} \tag{2.12}$$

Combining (2.11) and (2.12), we get (2.10). The proof is complete. \square

For $h \in \text{DRI}_c^+(\mathbb{R})$, define

$$\epsilon_t(h) := \sup_{x \in \mathbb{R}} \left| \sqrt{t} \mathbf{E}_x(h(\xi_t)) - \ell(h) \phi\left(\frac{x}{\sqrt{t}}\right) \right| \quad \text{and} \quad \tilde{\epsilon}_t(h) := \sup_{q>t} \epsilon_q(h). \tag{2.13}$$

By the definition we easily see that, for any $t > 0$, $\epsilon_t(h) \leq \sqrt{t} \|h\|_\infty + \frac{\ell(h)}{\sqrt{2\pi}}$. By Lemma 2.4, we have $\sup_{t>0} \epsilon_t(h) + \sup_{t>0} \tilde{\epsilon}_t(h) < \infty$ and

$$\lim_{t \rightarrow \infty} \tilde{\epsilon}_t(h) = \lim_{t \rightarrow \infty} \epsilon_t(h) = 0. \tag{2.14}$$

Since $\ell(h) \phi(xt^{-1/2}) \leq \frac{\ell(h)}{\sqrt{2\pi}}$ for any $x \in \mathbb{R}$, we have that, for any $h \in \text{DRI}_c^+(\mathbb{R})$ and $g \in B_{Lip}^+(\mathbb{R})$,

$$C(g, h) := \|g\|_\infty + \sup_{x \in \mathbb{R}, t>0} \sqrt{t} \mathbf{E}_x(h(\xi_t)) < \infty. \tag{2.15}$$

Lemma 2.5. Assume that (H2), (H3) and (H4) hold. Let $h \in \text{DRI}_c^+(\mathbb{R})$ and $g \in B_{Lip}^+(\mathbb{R})$.

(i) For any $t, r > 0$ and $y, z \in \mathbb{R}$, we have

$$\sqrt{t} \left| \mathbf{E}_{\sqrt{ty}}(h(\xi_{tr})) - \mathbf{E}_{\sqrt{tz}}(h(\xi_{tr})) \right| \lesssim \frac{\epsilon_{tr}(h)}{\sqrt{r}} + \frac{\sqrt{|y-z|}}{r^{3/4}}$$

and

$$\left| \mathbf{E}_{\sqrt{ty}} \left(g \left(\frac{\xi_{tr}}{\sqrt{t}} \right) \right) - \mathbf{E}_{\sqrt{tz}} \left(g \left(\frac{\xi_{tr}}{\sqrt{t}} \right) \right) \right| \lesssim |y-z|.$$

(ii) For any $t > 0, 0 < r < s$ with $s-r \in (0, 1)$ and $y \in \mathbb{R}$, we have

$$\sqrt{t} \left| \mathbf{E}_{\sqrt{ty}}(h(\xi_{tr})) - \mathbf{E}_{\sqrt{ty}}(h(\xi_{ts})) \right| \leq G_h(r, s; t),$$

where

$$G_h(r, s; t) := \frac{\epsilon_{tr}(h)}{\sqrt{r}} + \frac{\epsilon_{ts}(h)}{\sqrt{s}} + \ell(h) \left(\frac{1}{\sqrt{r}} - \frac{1}{\sqrt{s}} + \frac{1}{\sqrt{s}} \left(\sqrt{s-r} + 1 - \exp \left\{ -\frac{\sqrt{s-r}}{r} \right\} \right) \right).$$

Furthermore, for any $t > 0, 0 < r < s$ and $y \in \mathbb{R}$, it holds that

$$\left| \mathbf{E}_{\sqrt{ty}} \left(g \left(\frac{\xi_{tr}}{\sqrt{t}} \right) \right) - \mathbf{E}_{\sqrt{ty}} \left(g \left(\frac{\xi_{ts}}{\sqrt{t}} \right) \right) \right| \lesssim \sqrt{s-r}.$$

Proof. (i) The second inequality follows easily from

$$\left| \mathbf{E}_{\sqrt{ty}} \left(g \left(\frac{\xi_{tr}}{\sqrt{t}} \right) \right) - \mathbf{E}_{\sqrt{tz}} \left(g \left(\frac{\xi_{tr}}{\sqrt{t}} \right) \right) \right| \leq \mathbf{E}_0 \left(\left| g \left(\frac{\xi_{tr}}{\sqrt{t}} + y \right) - g \left(\frac{\xi_{tr}}{\sqrt{t}} + z \right) \right| \right) \leq \text{Lip}(g) |y-z| \lesssim |y-z|.$$

Now we prove the first inequality. By the definition of $\epsilon_t(h)$, we have

$$\sqrt{t} \left| \mathbf{E}_{\sqrt{ty}}(h(\xi_{tr})) - \mathbf{E}_{\sqrt{tz}}(h(\xi_{tr})) \right| \leq \frac{2\epsilon_{tr}(h)}{\sqrt{r}} + \frac{\ell(h)}{\sqrt{r}} \left| \phi\left(\frac{y}{\sqrt{r}}\right) - \phi\left(\frac{z}{\sqrt{r}}\right) \right|. \tag{2.16}$$

Applying Lemma 2.1(i) and (2.16), we get the first inequality in (i).

(ii) By Hölder’s inequality, we have

$$\begin{aligned} \left| \mathbf{E}_{\sqrt{ty}} \left(g \left(\frac{\xi_{tr}}{\sqrt{t}} \right) \right) - \mathbf{E}_{\sqrt{ty}} \left(g \left(\frac{\xi_{ts}}{\sqrt{t}} \right) \right) \right| &\leq \mathbf{E}_{\sqrt{ty}} \left(\left| g \left(\frac{\xi_{tr}}{\sqrt{t}} \right) - g \left(\frac{\xi_{ts}}{\sqrt{t}} \right) \right| \right) \\ &\leq \text{Lip}(g) \mathbf{E}_{\sqrt{ty}} \left(\left| \frac{\xi_{tr}}{\sqrt{t}} - \frac{\xi_{ts}}{\sqrt{t}} \right| \right) \lesssim \sqrt{\mathbf{E}_{\sqrt{ty}} \left(\left| \frac{\xi_{tr}}{\sqrt{t}} - \frac{\xi_{ts}}{\sqrt{t}} \right|^2 \right)} = \sqrt{s-r}. \end{aligned}$$

The second inequality follows immediately. Now we prove the first inequality of (ii). Combining Lemma 2.1 (ii) and (2.16),

$$\sqrt{t} \left| \mathbf{E}_{\sqrt{ty}}(h(\xi_{tr})) - \mathbf{E}_{\sqrt{ty}}(h(\xi_{ts})) \right| \leq \frac{\epsilon_{tr}(h)}{\sqrt{r}} + \frac{\epsilon_{ts}(h)}{\sqrt{s}} + \ell(h) \left| \frac{1}{\sqrt{r}} \phi\left(\frac{y}{\sqrt{r}}\right) - \frac{1}{\sqrt{s}} \phi\left(\frac{y}{\sqrt{s}}\right) \right|$$

$$\leq \frac{\epsilon_{tr}(h)}{\sqrt{r}} + \frac{\epsilon_{ts}(h)}{\sqrt{s}} + \ell(h) \left(\frac{1}{\sqrt{r}} - \frac{1}{\sqrt{s}} + \frac{1}{\sqrt{s}} \left(\sqrt{s-r} + 1 - \exp \left\{ -\frac{\sqrt{s-r}}{r} \right\} \right) \right),$$

which implies the desired result. \square

For $x \in \mathbb{R}$, define

$$\tau_x^+ := \inf\{t > 0 : \xi_t \geq x\}, \quad \tau_x^- := \inf\{t > 0 : \xi_t \leq x\}.$$

The following result on the overshoot of ξ is proved in [16, Lemma 2.1].

Lemma 2.6. *Assume that (H2) holds.*

(i) *If $\mathbf{E}_0(((-\xi_1) \vee 0)^\lambda) < \infty$ for some $\lambda > 2$, then*

$$\sup_{x>0} \mathbf{E}_x \left(\left| \xi_{\tau_0^-} \right|^{\lambda-2} \right) < \infty.$$

(ii) *If $\mathbf{E}_0((\xi_1 \vee 0)^\lambda) < \infty$ for some $\lambda > 2$, then*

$$\sup_{x>0} \mathbf{E}_{-x} \left(\xi_{\tau_0^+}^{\lambda-2} \right) < \infty.$$

2.3. Evolution equation for $(Z_t)_{t \geq 0}$

In this section, we always assume that (H1)–(H4) hold.

For any $f \in B^+(\mathbb{R})$, define

$$v_f(t, y) := 1 - \mathbf{E}_y \left(\exp \left\{ - \int_{\mathbb{R}} f(x) Z_t(dx) \right\} \right).$$

The next lemma gives an integral equation for v_f . Using [26, Lemma 4.1], its proof is standard and similar to that in [21, Lemma 2.1], and so we omit it. Define

$$\psi(v) := \beta \left(\sum_{k=0}^{\infty} p_k (1-v)^k - (1-v) \right), \quad v \in [0, 1].$$

Since $\sum_{k=0}^{\infty} k p_k = 1$, by Jensen’s inequality, we have that

$$\psi(v) \geq \beta \left((1-v)^{\sum_{k=0}^{\infty} k p_k} - (1-v) \right) = 0.$$

Lemma 2.7. *For any $t > 0, y \in \mathbb{R}$, $v_f(t, y)$ solves the equation*

$$v_f(t, y) = \mathbf{E}_y \left(1 - e^{-f(\xi_t)} \right) - \mathbf{E}_y \left(\int_0^t \psi \left(v_f(t-s, \xi_s) \right) ds \right).$$

For any $t, r > 0$, we define

$$\psi^{(t)}(v) := t^{\frac{\alpha}{\alpha-1}} \psi \left(vt^{\frac{1}{\alpha-1}} \right), \quad \xi_r^{(t)} := \frac{\xi_{tr}}{\sqrt{t}}. \tag{2.17}$$

For any $h \in C_c^+(\mathbb{R}), g \in B_{Lip}^+(\mathbb{R}), t, r > 0$ and $y \in \mathbb{R}$, we define

$$f^{(t)}(\cdot) := \frac{1}{t^{\frac{1}{\alpha-1} - \frac{1}{2}}} h(\cdot) + \frac{1}{t^{\frac{1}{\alpha-1}}} g \left(\frac{\cdot}{\sqrt{t}} \right), \quad v_{g,h}^{(t)}(r, y) := t^{\frac{1}{\alpha-1}} v_{f^{(t)}}(tr, \sqrt{t}y). \tag{2.18}$$

With a slight abuse of notation, we will use the same notation \mathbf{P}_y to denote the law of $\xi_r^{(t)}$ starting from $\xi_0^{(t)} = y$, i.e.,

$$(\xi_r^{(t)}, \mathbf{P}_y) := \left(\frac{1}{\sqrt{t}} \xi_{tr}, \mathbf{P}_{\sqrt{t}y} \right). \tag{2.19}$$

It is well-known (for example, see [21, Lemma 2.14(ii)]) that, under (H1), for any $K > 0$,

$$\lim_{t \rightarrow \infty} \psi^{(t)}(v) = C(\alpha)v^\alpha \quad \text{uniformly for } v \in [0, K]. \tag{2.20}$$

Lemma 2.8. *There exists a constant $C_\psi \in (0, \infty)$ such that*

$$\left| \psi^{(t)}(u) - \psi^{(t)}(v) \right| \leq C_\psi (u^{\alpha-1} + v^{\alpha-1}) |u - v|, \quad \forall u, v \in [0, t^{\frac{1}{\alpha-1}}], \quad \forall t > 0.$$

In particular, we have

$$\psi^{(t)}(v) \leq C_\psi v^\alpha, \quad \forall v \in [0, t^{\frac{1}{\alpha-1}}], \quad \forall t > 0.$$

Proof. We first prove that there exists some constant C_ψ such that

$$|\psi'(v)| \leq C_\psi v^{\alpha-1}, \quad \forall v \in [0, 1]. \tag{2.21}$$

First, using $\sum_{k=1}^\infty k p_k = 1$, we have

$$\begin{aligned} |\psi'(v)| &= \beta \left(1 - \sum_{k=1}^\infty k p_k (1-v)^{k-1} \right) = \beta v \sum_{k=2}^\infty k p_k \left(\sum_{j=0}^{k-2} (1-v)^j \right) \\ &= \beta v \sum_{j=0}^\infty (1-v)^j \sum_{k=j+2}^\infty k p_k. \end{aligned} \tag{2.22}$$

Under **(H1)**, we have $\sum_{k=n}^\infty p_k \lesssim n^{-\alpha}$ for all $n \geq 2$. Thus, for all $j \geq 0$,

$$\begin{aligned} \sum_{k=j+2}^\infty k p_k &= (j+1) \sum_{k=j+2}^\infty p_k + \sum_{k=j+2}^\infty \sum_{n=k}^\infty p_n \\ &\lesssim \frac{j+1}{(j+2)^\alpha} + \sum_{k=j+2}^\infty \frac{1}{k^\alpha} \lesssim \sum_{k=j+2}^\infty \frac{1}{k^\alpha}. \end{aligned} \tag{2.23}$$

Combining (2.22) and (2.23), we conclude that for all $v \in [0, 1]$,

$$|\psi'(v)| \lesssim v \sum_{j=0}^\infty (1-v)^j \sum_{k=j+2}^\infty \frac{1}{k^\alpha} = \sum_{k=2}^\infty \frac{1}{k^\alpha} (1 - (1-v)^{k-1}).$$

Together with the inequality $1 - (1-v)^{k-1} \leq 1 \wedge ((k-1)v)$, we obtain that for all $v \in [0, 1]$,

$$|\psi'(v)| \lesssim \sum_{k=2}^\infty \frac{1}{k^\alpha} (1 \wedge ((k-1)v)) \leq \int_1^\infty \frac{1}{x^\alpha} (1 \wedge (xv)) dx = v \int_1^{1/v} \frac{1}{x^{\alpha-1}} dx + \int_{1/v}^\infty \frac{1}{x^\alpha} dx,$$

which implies (2.21).

Now we assume that $u < v$. Then there exists $\eta \in [u, v]$ such that

$$\begin{aligned} |\psi^{(t)}(u) - \psi^{(t)}(v)| &= t^{\frac{\alpha}{\alpha-1}} \left| \psi(u t^{-\frac{1}{\alpha-1}}) - \psi(v t^{-\frac{1}{\alpha-1}}) \right| \\ &= t |u - v| \left| \psi'(\eta t^{-\frac{1}{\alpha-1}}) \right| \leq C_\psi \eta^{\alpha-1} |u - v| \leq C_\psi v^{\alpha-1} |u - v|, \end{aligned}$$

where in the second to last inequality, we used (2.21). The proof is complete. \square

Lemma 2.9. For any $h \in \text{DRI}_c^+(\mathbb{R})$ and $g \in B_{Lip}^+(\mathbb{R})$, it holds that

$$\begin{aligned} v_{g,h}^{(t)}(r, y) &= t^{\frac{1}{\alpha-1}} \mathbf{E}_y \left(1 - \exp \left\{ -\frac{1}{t^{\frac{1}{\alpha-1} - \frac{1}{2}}} h(\sqrt{t} \xi_r^{(t)}) - \frac{1}{t^{\frac{1}{\alpha-1}}} g(\xi_r^{(t)}) \right\} \right) \\ &\quad - \mathbf{E}_y \left(\int_0^r \psi^{(t)} \left(v_{g,h}^{(t)}(r-s, \xi_s^{(t)}) \right) ds \right). \end{aligned}$$

Proof. Combining (2.17), (2.18) and Lemma 2.7, we get that

$$\begin{aligned} v_{g,h}^{(t)}(r, y) &= t^{\frac{1}{\alpha-1}} \mathbf{E}_y \left(1 - \exp \left\{ -\frac{1}{t^{\frac{1}{\alpha-1} - \frac{1}{2}}} h(\sqrt{t} \xi_r^{(t)}) - \frac{1}{t^{\frac{1}{\alpha-1}}} g(\xi_r^{(t)}) \right\} \right) \\ &\quad - t^{\frac{1}{\alpha-1}} \mathbf{E}_{\sqrt{ty}} \left(\int_0^{tr} \psi \left(v_{g,h}^{(t)}(tr-s, \xi_s) \right) ds \right) \\ &= t^{\frac{1}{\alpha-1}} \mathbf{E}_y \left(1 - \exp \left\{ -f^{(t)}(\sqrt{t} \xi_r^{(t)}) \right\} \right) - t^{\frac{1}{\alpha-1}} \mathbf{E}_y \left(\int_0^{tr} \psi \left(v_{g,h}^{(t)}(tr-s, \sqrt{t} \xi_{s/t}^{(t)}) \right) ds \right) \\ &= t^{\frac{1}{\alpha-1}} \mathbf{E}_y \left(1 - \exp \left\{ -f^{(t)}(\sqrt{t} \xi_r^{(t)}) \right\} \right) - t^{\frac{\alpha}{\alpha-1}} \mathbf{E}_y \left(\int_0^r \psi \left(v_{g,h}^{(t)}(tr-ts, \sqrt{t} \xi_s^{(t)}) \right) ds \right) \\ &= t^{\frac{1}{\alpha-1}} \mathbf{E}_y \left(1 - \exp \left\{ -f^{(t)}(\sqrt{t} \xi_r^{(t)}) \right\} \right) - t^{\frac{\alpha}{\alpha-1}} \mathbf{E}_y \left(\int_0^r \psi \left(t^{-\frac{1}{\alpha-1}} v_{g,h}^{(t)}(r-s, \xi_s^{(t)}) \right) ds \right). \end{aligned}$$

The desired result now follows immediately from the definition of $\psi^{(t)}$. \square

2.4. Initial trace theory

For any open set $U \subset \mathbb{R}$, we denote by $C_c^+(U)$ the family of non-negative continuous functions with compact support in U . Denote by $\mathcal{B}_{reg}^+(\mathbb{R})$ the space of positive outer regular Borel measures. Suppose that $\Lambda \subset \mathbb{R}$ is a closed set and that η is a non-negative Radon

measure on Λ^c . By [27, pp.1452–1453], the pair (Λ, η) can be represented by the following measure $\gamma_{(\Lambda, \eta)} \in \mathcal{B}_{reg}^+(\mathbb{R})$:

$$\gamma_{(\Lambda, \eta)}(B) := \begin{cases} \infty, & B \cap \Lambda \neq \emptyset, \\ \eta(B), & B \cap \Lambda = \emptyset. \end{cases}$$

Define the set of regular points of $\gamma_{(\Lambda, \eta)}$ by

$$\mathcal{R}_{(\Lambda, \eta)} := \{x \in \mathbb{R} : \gamma_{(\Lambda, \eta)}((x - z, x + z)) = \infty, \quad \forall z > 0\}^c.$$

For any closed set $\hat{\Lambda} \subset \mathbb{R}$ and non-negative Radon measure $\hat{\eta}$ on $\hat{\Lambda}^c$, consider the problem

$$\begin{cases} \frac{\partial}{\partial r} \hat{v}_{(\hat{\Lambda}, \hat{\eta})}^X(r, y) = \frac{\partial^2}{\partial y^2} \hat{v}_{(\hat{\Lambda}, \hat{\eta})}^X(r, y) - \left(\hat{v}_{(\hat{\Lambda}, \hat{\eta})}^X(r, y)\right)^\alpha, & (r, y) \in (0, \infty) \times \mathbb{R}, \\ \left\{x \in \mathbb{R} : \forall z > 0, \lim_{r \downarrow 0} \int_{x-z}^{x+z} \hat{v}_{(\hat{\Lambda}, \hat{\eta})}^X(r, y) dy = \infty\right\} = \hat{\Lambda}, \\ \forall f \in C_c^+(\hat{\Lambda}^c), \lim_{r \downarrow 0} \int f(y) \hat{v}_{(\hat{\Lambda}, \hat{\eta})}^X(r, y) dy = \int f(y) \hat{\eta}(dy). \end{cases} \tag{2.24}$$

Define $\Lambda := \{x/\sqrt{2} : x \in \hat{\Lambda}\}$ and let η be the Radon measure on Λ^c such that

$$\int_{\Lambda^c} f(y) \eta(dy) := \frac{1}{\sqrt{2} C(\alpha)^{\frac{1}{\alpha-1}}} \int_{\hat{\Lambda}^c} f(\sqrt{2}y) \hat{\eta}(dy).$$

Consider the problem

$$\begin{cases} \frac{\partial}{\partial r} v_{(\Lambda, \eta)}^X(r, y) = \frac{1}{2} \frac{\partial^2}{\partial y^2} v_{(\Lambda, \eta)}^X(r, y) - \varphi\left(v_{(\Lambda, \eta)}^X(r, y)\right), & (r, y) \in (0, \infty) \times \mathbb{R}, \\ \left\{x \in \mathbb{R} : \forall z > 0, \lim_{r \downarrow 0} \int_{x-z}^{x+z} v_{(\Lambda, \eta)}^X(r, y) dy = \infty\right\} = \Lambda, \\ \forall f \in C_c^+(\Lambda^c), \lim_{r \downarrow 0} \int f(y) v_{(\Lambda, \eta)}^X(r, y) dy = \int f(y) \eta(dy). \end{cases} \tag{2.25}$$

It is easy to check that

$$v_{(\Lambda, \eta)}^X(r, y) = \frac{1}{C(\alpha)^{\frac{1}{\alpha-1}}} \hat{v}_{(\hat{\Lambda}, \hat{\eta})}^X\left(r, \frac{y}{\sqrt{2}}\right)$$

is a one-to-one correspondence between the positive solutions of (2.24) and the positive solutions of (2.25). According to [27, Theorem 3.5], (2.24) has a unique positive solution $\hat{v}_{(\hat{\Lambda}, \hat{\eta})}^X(r, y)$. Consequently, the function $v_{(\Lambda, \eta)}^X(r, y)$ defined above is the unique solution of (2.25). We call (Λ, η) the initial trace of the solution $v_{(\Lambda, \eta)}^X$.

In this section, we give a probabilistic representation of the solution $v_{(\Lambda, \eta)}^X$. To avoid too much measure theoretic details, we only deal with the case when Λ is a bounded closed interval. In the special case $\varphi(\lambda) = \frac{1}{2}\lambda^2$, a probabilistic representation via Brownian snake was given by Le Gall [28, Theorem 4].

Recall that X is a critical super-Brownian motion with branching mechanism φ given in (1.12) and $\{Y_t(x) : t > 0, x \in \mathbb{R}\}$ is the density process of X which, for all $y \in \mathbb{R}$, is \mathbb{P}_{δ_y} -almost surely continuous in x for all $t > 0$. Let $v_{(\Lambda, \eta)}^X$ is the solution of the PDE problem (2.25).

Proposition 2.10. *Suppose that $\Lambda = [a, b] \subset \mathbb{R}$ is a bounded closed interval and η is a Radon measure on Λ^c . Then for any $r > 0, y \in \mathbb{R}$,*

$$v_{(\Lambda, \eta)}^X(r, y) = -\log \mathbb{E}_{\delta_y} \left(1_{\{Y_r(x)=0, \forall x \in \Lambda\}} e^{-\int Y_r(z) \eta(dz)} \right). \tag{2.26}$$

Before presenting the proof, we first recall the notion of m -weak convergence from [27, Definition 3.9].

Definition 2.11. Let (Λ_n, η_n) be a sequence of initial traces and (Λ, η) be another initial trace. We say that the measures $\gamma_{(\Lambda_n, \eta_n)}$ converge m -weakly to the measure $\gamma_{(\Lambda, \eta)}$ if the following two conditions hold:

- (i) If $U \subset \mathbb{R}$ is an open set with $\gamma_{(\Lambda, \eta)}(U) = \infty$, then $\lim_{n \rightarrow \infty} \gamma_{(\Lambda_n, \eta_n)}(U) = \infty$.
- (ii) For any compact set $K \subset \mathcal{R}_{(\Lambda, \eta)}$, the sequence of $\gamma_{(\Lambda_n, \eta_n)}(K)$ is eventually bounded, i.e., there exists $N \in \mathbb{N}$ and $C \in (0, \infty)$ such that $\gamma_{(\Lambda_n, \eta_n)}(K) \leq C$ for all $n \geq N$, and for any $\phi \in C_c^+(\mathcal{R}_{(\Lambda, \eta)})$, $\lim_{n \rightarrow \infty} \int \phi(x) \gamma_{(\Lambda_n, \eta_n)}(dx) = \int \phi(x) \gamma_{(\Lambda, \eta)}(dx)$.

According to [29, Section 2.1], if $\gamma_{(\Lambda_n, \eta_n)}$ converges m -weakly to $\gamma_{(\Lambda, \eta)}$, then for any $r > 0$ and $y \in \mathbb{R}$, $v_{(\Lambda_n, \eta_n)}^X(r, y)$ converges to $v_{(\Lambda, \eta)}^X(r, y)$ as $n \rightarrow \infty$. Now we are ready to prove Proposition 2.10.

Proof of Proposition 2.10. Step 1: In this part we consider the case that $\Lambda = \emptyset$. It is well-known that for any $r > 0$, $Y_r(\cdot)$ is compactly supported (to see this, one can fix x and t and let the constant Λ in [30, Lemma 4.3] tend to ∞). Therefore, by the Markov property and the dominated convergence theorem,

$$v_{(\emptyset, \eta)}^X(r, y) = \lim_{s \downarrow 0} v_{(\emptyset, \eta)}^X(r + s, y) = -\lim_{s \downarrow 0} \log \mathbb{E}_{\delta_y} \left(\exp \left\{ -\int v_{(\emptyset, \eta)}^X(s, z) Y_r(z) dz \right\} \right)$$

$$= -\log \mathbb{E}_{\delta_y} \left(\lim_{s \downarrow 0} \exp \left\{ - \int v_{(\emptyset, \eta)}^X(s, z) Y_r(z) dz \right\} \right) = -\log \mathbb{E}_{\delta_y} \left(\exp \left\{ - \int Y_r(z) \eta(dz) \right\} \right),$$

which implies (2.26) in the case $\Lambda = \emptyset$.

Step 2: In this step we consider the case that $\Lambda \subset \mathbb{R}$ is a closed subset and that η is a Radon measure on Λ^c . Define $\eta_\Lambda(dx) = 1_\Lambda dx$ if $\ell(\Lambda) = b - a \neq 0$ and $\eta_\Lambda(dx) = \delta_a(dx)$ if $\Lambda = \{a\}$. For each n , define

$$A_n := \emptyset, \quad B_n := \left\{ y \in \mathbb{R} : \text{dist}(y, \Lambda) \leq \frac{1}{n} \right\} \quad \text{and} \quad \eta_n := n\eta_\Lambda + \eta|_{B_n^c},$$

then for any n , η_n is a Radon measure on \mathbb{R} . By the result obtained in Step 1, we have

$$\begin{aligned} v_{(A_n, \eta_n)}^X(r, y) &= -\log \mathbb{E}_{\delta_y} \left(\exp \left\{ - \int Y_r(z) \eta_n(dz) \right\} \right) \\ &= -\log \mathbb{E}_{\delta_y} \left(\exp \left\{ -n \int_\Lambda Y_r(z) \eta_\Lambda(dz) - \int_{B_n^c} Y_r(z) \eta(dz) \right\} \right). \end{aligned} \tag{2.27}$$

Since $B_n^c \uparrow \Lambda^c$ as $n \rightarrow \infty$, combining the dominated convergence theorem and (2.27), we see that

$$\begin{aligned} \lim_{n \rightarrow \infty} v_{(A_n, \eta_n)}^X(r, y) &= -\log \mathbb{E}_{\delta_y} \left(1_{\{\int_\Lambda Y_r(z) \eta_\Lambda(z) = 0\}} \exp \left\{ - \int_{\Lambda^c} Y_r(z) \eta(dz) \right\} \right) \\ &= -\log \mathbb{E}_{\delta_y} \left(1_{\{Y_r(z) = 0, \forall z \in \Lambda\}} \exp \left\{ - \int Y_r(z) \eta(dz) \right\} \right), \end{aligned}$$

where in the last equality we used the fact that the support of η_Λ is equal to Λ and that the support of η is a subset of Λ^c . Therefore, to complete the proof, it suffices to show that (A_n, η_n) converges m -weakly to (Λ, η) . Now we check the conditions in Definition 2.11. For (i), suppose that $\gamma_{(\Lambda, \eta)}(U) = \infty$ for an open set $U \subset \mathbb{R}$. If $U \cap \Lambda \neq \emptyset$, then by the definition of η_Λ , we can find a Borel set $B \subset U \cap \Lambda$ such that $\eta_\Lambda(B) > 0$. In this case,

$$\eta_n(U) \geq \eta_n(B) = n\eta_\Lambda(B) \xrightarrow{n \rightarrow \infty} \infty.$$

On the other hand, if $U \cap \Lambda = \emptyset$, then

$$\eta(U) = \infty = \eta(U \cap \Lambda^c) = \lim_{n \rightarrow \infty} \eta(U \cap B_n^c) \leq \lim_{n \rightarrow \infty} \eta_n(U),$$

as desired. Now we check (ii). Note that $\mathcal{R}_{(\Lambda, \eta)} = \Lambda^c$. For any compact set $K \subset \Lambda^c$, $\eta_n(K) = \eta(K \cap B_n^c) \leq \eta(K) < \infty$ is bounded. Besides, for any $\phi \in C_c^+(\Lambda^c)$, by the monotone convergence theorem,

$$\lim_{n \rightarrow \infty} \int \phi(x) \gamma_{(\Lambda_n, \eta_n)}(dx) = \lim_{n \rightarrow \infty} \int_{B_n^c} \phi(x) \eta(dx) = \int_{\Lambda^c} \phi(x) \eta(dx) = \int \phi(x) \gamma_{(\Lambda, \eta)}(dx),$$

which implies (ii). This completes the proof of the proposition. \square

Remark 2.12. We will need the following result later: for any $r > 0, y \in \mathbb{R}$,

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} v_{([-\varepsilon, \varepsilon], 0)}^X(r, y) &= -\lim_{\varepsilon \rightarrow 0} \log \mathbb{E}_{\delta_y} (Y_r(x) = 0, \forall x \in [-\varepsilon, \varepsilon]) \\ &= v_{(\{0\}, 0)}^X(r, y) = -\log \mathbb{E}_{\delta_y} (Y_r(0) = 0). \end{aligned} \tag{2.28}$$

To prove (2.28), we only need to show that $\gamma_{([-\varepsilon, \varepsilon], 0)}$ converges m -weakly to $\gamma_{(\{0\}, 0)}$. Condition (i) of Definition 2.11 is easy to check since $\gamma_{([-\varepsilon, \varepsilon], 0)}(U) \geq \gamma_{(\{0\}, 0)}(U)$. For (ii), for compact set $K \subset \mathcal{R}_{(\{0\}, 0)} = \mathbb{R} \setminus \{0\}$, let $\varepsilon_0 > 0$ sufficient small so that $K \subset [-\varepsilon_0, \varepsilon_0]^c$. Then $\gamma_{([-\varepsilon, \varepsilon], 0)}(K) = 0$ when $\varepsilon < \varepsilon_0$. Furthermore, for any $\phi \in C_c^+(\mathbb{R} \setminus \{0\})$, suppose that the support of ϕ is a subset of $[-\varepsilon_0, \varepsilon_0]^c$, then for any $\varepsilon < \varepsilon_0$, it holds that $\int \phi(x) \gamma_{([-\varepsilon, \varepsilon], 0)}(dx) = 0 = \int \phi(x) \gamma_{(\{0\}, 0)}(dx)$. Hence (2.28) is true.

3. Proof of the main results

In this section, we always assume that (H1)–(H4) hold.

Lemma 3.1. Let $h \in \text{DRI}_c^+(\mathbb{R})$ and $g \in B_{Lip}^+(\mathbb{R})$, and let $v_{g,h}^{(t)}$ be given in (2.18). Suppose $r > 0$. Then there exists a constant $C_1 = C_1(g, h)$ such that for any $t > 1, y \in \mathbb{R}$ and $s \in [0, r]$,

$$\mathbb{E}_y \left(\left(v_{g,h}^{(t)}(r-s, \xi_s^{(t)}) \right)^{\alpha-1} \right) \leq \left(\mathbb{E}_y \left(v_{g,h}^{(t)}(r-s, \xi_s^{(t)}) \right) \right)^{\alpha-1} \leq C_1 \left(\frac{1}{r^{(\alpha-1)/2}} \wedge t^{(\alpha-1)/2} \right) \tag{3.1}$$

and that

$$\mathbb{E}_y \left(\psi^{(t)} \left(v_{g,h}^{(t)}(r-s, \xi_s^{(t)}) \right) \right) \leq \frac{C_1}{(r-s)^{(\alpha-1)/2} \sqrt{r}}. \tag{3.2}$$

Proof. The first inequality of (3.1) follows directly from Jensen’s inequality. Now we prove the second inequality of (3.1). Combining Lemma 2.9, (2.15), (2.19) and the fact that $1 - e^{-|x|} \leq |x|$, we get that for all $t > 1, y \in \mathbb{R}$ and $r > 0$,

$$\begin{aligned} v_{g,h}^{(t)}(r, y) &\leq \sqrt{t} \mathbf{E}_y \left(h(\sqrt{t} \xi_r^{(t)}) \right) + \mathbf{E}_y \left(g(\xi_r^{(t)}) \right) \\ &\leq \left(\frac{C(g, h)}{\sqrt{r}} \right) \wedge (\sqrt{t} \|h\|_\infty + \|g\|_\infty) \lesssim \frac{1}{\sqrt{r}} \wedge \sqrt{t}. \end{aligned} \tag{3.3}$$

Therefore, combining (3.3) and the Markov property, for any $t > 1, y \in \mathbb{R}, s \in [0, r]$,

$$\begin{aligned} \mathbf{E}_y \left(v_{g,h}^{(t)}(r - s, \xi_s^{(t)}) \right) &\leq \mathbf{E}_y \left(\sqrt{t} \mathbf{E}_{\xi_s^{(t)}} \left(h(\sqrt{t} \xi_{r-s}^{(t)}) \right) + \mathbf{E}_{\xi_s^{(t)}} \left(g(\xi_{r-s}^{(t)}) \right) \right) \\ &= \sqrt{t} \mathbf{E}_y \left(h(\sqrt{t} \xi_r^{(t)}) \right) + \mathbf{E}_y \left(g(\xi_r^{(t)}) \right) \lesssim \frac{1}{\sqrt{r}} \wedge \sqrt{t}, \end{aligned}$$

which implies (3.1). For (3.2), combining Lemma 2.8, (3.1) and (3.3),

$$\begin{aligned} \mathbf{E}_y \left(\psi^{(t)} \left(v_{g,h}^{(t)}(r - s, \xi_s^{(t)}) \right) \right) &\lesssim \mathbf{E}_y \left(\left(v_{g,h}^{(t)}(r - s, \xi_s^{(t)}) \right)^\alpha \right) \\ &\lesssim \frac{1}{(r - s)^{(\alpha-1)/2}} \mathbf{E}_y \left(v_{g,h}^{(t)}(r - s, \xi_s^{(t)}) \right) \lesssim \frac{1}{(r - s)^{(\alpha-1)/2} \sqrt{r}}. \end{aligned}$$

The proof is now complete. \square

Recall the definition of $\tilde{\epsilon}_t(h)$ defined in (2.13).

Proposition 3.2. Assume $h \in \text{DRI}_c^+(\mathbb{R}), g \in B_{Lip}^+(\mathbb{R})$ and $T > 0$. Let $v_{g,h}^{(t)}$ be defined as in (2.18).

(i) There exists a constant $N_1 = N_1(g, h, T, \psi) > 0$ such that for all $t > 1, r \in (0, T]$ and $y, z \in \mathbb{R}$ with $|y - z| < 1$,

$$\left| v_{g,h}^{(t)}(r, y) - v_{g,h}^{(t)}(r, z) \right| \leq \frac{N_1}{r^{3/4}} \left(\tilde{\epsilon}_{\sqrt{tr}}(h) + \sqrt{|y - z|} + \frac{1}{t^{1/4}} \right). \tag{3.4}$$

(ii) There exists a constant $N_2 = N_2(g, h, T, \psi) > 0$ such that for all $t > 1, r \in (0, T], q \in (0, 1)$ and $y \in \mathbb{R}$,

$$\left| v_{g,h}^{(t)}(r, y) - v_{g,h}^{(t)}(r + q, y) \right| \leq \frac{N_2}{r^{3/2}} \left(\tilde{\epsilon}_{\sqrt{tr}}(h) + q^{1/8} + \frac{1}{t^{1/4}} \right). \tag{3.5}$$

Proof. (i) Without loss of generality, we assume that $y < z$. Combining (2.4) with Lemmas 2.8 and 2.9, we get that for all $t > 1, y, z \in \mathbb{R}$ and $r \in (0, T]$,

$$\begin{aligned} &\left| v_{g,h}^{(t)}(r, y) - v_{g,h}^{(t)}(r, z) \right| \\ &\leq \frac{t^{\frac{1}{\alpha-1}}}{t^{\frac{2}{\alpha-1}-1}} \left(\mathbf{E}_y \left(h^2(\sqrt{t} \xi_r^{(t)}) \right) + \mathbf{E}_z \left(h^2(\sqrt{t} \xi_r^{(t)}) \right) \right) + \frac{1}{t^{\frac{1}{\alpha-1}}} \left(\mathbf{E}_y \left(g^2(\xi_r^{(t)}) \right) + \mathbf{E}_z \left(g^2(\xi_r^{(t)}) \right) \right) \\ &\quad + \sqrt{t} \sup_{x \in \mathbb{R}} \left| \mathbf{E}_x \left(h(\sqrt{t} \xi_r^{(t)}) \right) - \mathbf{E}_{x+y-z} \left(h(\sqrt{t} \xi_r^{(t)}) \right) \right| + \sup_{x \in \mathbb{R}} \left| \mathbf{E}_x \left(g(\xi_r^{(t)}) \right) - \mathbf{E}_{x+y-z} \left(g(\xi_r^{(t)}) \right) \right| \\ &\quad + C_\psi \int_0^r \left(\mathbf{E}_y \left(\left(v_{g,h}^{(t)}(r - s, \xi_s^{(t)}) \right)^{\alpha-1} \right) + \mathbf{E}_z \left(\left(v_{g,h}^{(t)}(r - s, \xi_s^{(t)}) \right)^{\alpha-1} \right) \right) \\ &\quad \times \sup_{x \in \mathbb{R}} \left| v_{g,h}^{(t)}(r - s, x) - v_{g,h}^{(t)}(r - s, x + z - y) \right| ds. \end{aligned}$$

Define

$$F_{z-y}^f(r) := \sup_{x \in \mathbb{R}} \left| v_{g,h}^{(t)}(r, x) - v_{g,h}^{(t)}(r, x + z - y) \right|.$$

Combining Lemma 2.5(i), (2.15), (2.19), Lemma 3.1 and the fact that $\frac{1}{\alpha-1} \geq 1$, we conclude from the above inequality that for any $y, z \in \mathbb{R}, r \in (0, T]$ and $t > 1$,

$$F_{z-y}^f(r) \lesssim \frac{1}{\sqrt{tr}} \wedge 1 + \frac{1}{t} + \left(\frac{\epsilon_{tr}(h)}{\sqrt{r}} + \frac{\sqrt{|y - z|}}{r^{3/4}} \right) \wedge \sqrt{t} + |y - z| + \frac{1}{r^{(\alpha-1)/2}} \int_0^r F_{z-y}^f(s) ds.$$

Define $G_{z-y}^f(r) := r^{(\alpha-1)/2} F_{z-y}^f(r)$. Then there exists a constant $K_1 = K_1(g, h, \psi, T) \in (0, \infty)$ such that for all $y, z \in \mathbb{R}$ with $|y - z| < 1, t > 1$ and $r \in (0, T]$,

$$\begin{aligned} G_{z-y}^f(r) &\leq K_1 r^{(\alpha-1)/2} \left(\frac{1}{\sqrt{tr}} \wedge 1 + \frac{1}{t} + \left(\frac{\epsilon_{tr}(h)}{\sqrt{r}} + \frac{\sqrt{|y - z|}}{r^{3/4}} \right) \wedge \sqrt{t} + |y - z| \right) \\ &\quad + K_1 \int_0^r \frac{1}{s^{(\alpha-1)/2}} G_{z-y}^f(s) ds \\ &=: \alpha_f(r) + K_1 \int_0^r \frac{1}{s^{(\alpha-1)/2}} G_{z-y}^f(s) ds. \end{aligned}$$

It follows then from Gronwall's inequality that for all $y, z \in \mathbb{R}$ with $|y - z| < 1, t > 1$ and $r \in (0, T]$,

$$G_{z-y}^f(r) \leq \alpha_f(r) + K_1 \int_0^r \exp \left\{ K_1 \int_s^r \frac{1}{q^{(\alpha-1)/2}} dq \right\} \frac{\alpha_f(s)}{s^{(\alpha-1)/2}} ds$$

$$\lesssim \alpha_f(r) + \int_0^r \frac{\alpha_f(s)}{s^{(\alpha-1)/2}} ds. \tag{3.6}$$

Note that

$$\alpha_f(r) \lesssim r^{(\alpha-1)/2} \left(\frac{1}{\sqrt{tr}} \wedge 1 + \frac{1}{t} + \left(\frac{\epsilon_{tr}(h)}{\sqrt{r}} + \frac{\sqrt{|y-z|}}{r^{3/4}} \right) \wedge \sqrt{t} + |y-z| \right)$$

$$\leq r^{(\alpha-1)/2} \left(\frac{1}{t^{1/4}\sqrt{r}} + \frac{\sqrt{T}}{t^{1/4}\sqrt{r}} + \frac{\epsilon_{tr}(h)}{\sqrt{r}} + \frac{\sqrt{|y-z|}}{r^{3/4}} + \frac{T^{3/4}\sqrt{|y-z|}}{r^{3/4}} \right)$$

$$\lesssim r^{(\alpha-1)/2} \left(\frac{1}{t^{1/4}\sqrt{r}} + \frac{\epsilon_{tr}(h)}{\sqrt{r}} + \frac{\sqrt{|y-z|}}{r^{3/4}} \right) \tag{3.7}$$

and

$$\int_0^r \frac{\epsilon_{ts}(h)}{\sqrt{s}} ds \leq \int_0^r \frac{\tilde{\epsilon}_{ts}(h)}{\sqrt{s}} ds \leq \sup_{t>0} \epsilon_t(h) \int_0^{r/\sqrt{t}} \frac{1}{\sqrt{s}} ds + \tilde{\epsilon}_{\sqrt{tr}}(h) \int_0^T \frac{1}{\sqrt{s}} ds$$

$$\lesssim \sqrt{T} \frac{1}{t^{1/4}} + \sqrt{T} \tilde{\epsilon}_{\sqrt{tr}}(h) \lesssim \frac{1}{t^{1/4}} + \tilde{\epsilon}_{\sqrt{tr}}(h). \tag{3.8}$$

Therefore, combining (3.6), (3.7), (3.8) and the fact that $\epsilon_{tr}(h) \leq \tilde{\epsilon}_{\sqrt{tr}}(h)$, we see that for all $y, z \in \mathbb{R}$ with $|y - z| < 1, t > 1$ and $r \in (0, T]$,

$$G_{z-y}^f(r) \lesssim r^{(\alpha-1)/2} \left(\frac{1}{t^{1/4}\sqrt{r}} + \frac{\epsilon_{tr}(h)}{\sqrt{r}} + \frac{\sqrt{|y-z|}}{r^{3/4}} \right) + \left(\int_0^r \frac{1}{t^{1/4}\sqrt{s}} ds + \int_0^r \frac{\epsilon_{ts}(h)}{\sqrt{s}} ds + \int_0^r \frac{\sqrt{|y-z|}}{s^{3/4}} ds \right)$$

$$\lesssim r^{(\alpha-1)/2} \cdot \frac{1}{r^{3/4}} \left(\frac{T^{1/4}}{t^{1/4}} + T^{1/4} \tilde{\epsilon}_{\sqrt{tr}}(h) + \sqrt{|y-z|} \right)$$

$$+ r^{(\alpha-1)/2} \cdot \frac{T^{(5-2\alpha)/4} C(T)}{r^{3/4}} \left(\frac{2\sqrt{T}}{t^{1/4}} + \left(\frac{1}{t^{1/4}} + \tilde{\epsilon}_{\sqrt{tr}}(h) \right) + 4T^{1/4} \sqrt{|y-z|} \right)$$

$$\lesssim r^{(\alpha-1)/2} \cdot \frac{1}{r^{3/4}} \left(\frac{1}{t^{1/4}} + \tilde{\epsilon}_{\sqrt{tr}}(h) + \sqrt{|y-z|} \right),$$

which implies (3.4).

Now we prove (ii). Define

$$\hat{F}_q^f(r) := \sup_{x \in \mathbb{R}} \left| v_{g,h}^{(t)}(r, x) - v_{g,h}^{(t)}(r+q, x) \right|.$$

Then combining (2.4), Lemmas 2.8–2.9, and $\frac{1}{\alpha-1} \geq 1$, we get that for all $y \in \mathbb{R}, q \in (0, 1), t > 1$ and $r \in (0, T]$,

$$\left| v_{g,h}^{(t)}(r, y) - v_{g,h}^{(t)}(r+q, y) \right| \leq \frac{t^{\frac{1}{\alpha-1}}}{t^{\frac{2}{\alpha-1}-1}} \left(\mathbf{E}_y \left(h^2(\sqrt{t}\xi_r^{(t)}) \right) + \mathbf{E}_y \left(h^2(\sqrt{t}\xi_{r+q}^{(t)}) \right) \right)$$

$$+ \frac{1}{t^{\frac{1}{\alpha-1}}} \left(\mathbf{E}_y \left(g^2(\xi_r^{(t)}) \right) + \mathbf{E}_y \left(g^2(\xi_{r+q}^{(t)}) \right) \right) + \sqrt{t} \sup_{x \in \mathbb{R}} \left| \mathbf{E}_x(h(\sqrt{t}\xi_r^{(t)})) - \mathbf{E}_x(h(\sqrt{t}\xi_{r+q}^{(t)})) \right|$$

$$+ \sup_{x \in \mathbb{R}} \left| \mathbf{E}_x(g(\xi_r^{(t)})) - \mathbf{E}_x(g(\xi_{r+q}^{(t)})) \right| + \mathbf{E}_y \left(\int_r^{r+q} \psi^{(t)} \left(v_{g,h}^{(t)}(r+q-s, \xi_s^{(t)}) \right) ds \right)$$

$$+ \left| \mathbf{E}_y \left(\int_0^r \psi^{(t)} \left(v_{g,h}^{(t)}(r-s, \xi_s^{(t)}) \right) ds \right) - \mathbf{E}_y \left(\int_0^r \psi^{(t)} \left(v_{g,h}^{(t)}(r+q-s, \xi_s^{(t)}) \right) ds \right) \right|$$

$$\leq \mathbf{E}_y \left(h^2(\sqrt{t}\xi_r^{(t)}) + \mathbf{E}_y \left(h^2(\sqrt{t}\xi_{r+q}^{(t)}) \right) \right) + \frac{2}{t} \|g\|_\infty^2 + \sqrt{t} \sup_{x \in \mathbb{R}} \left| \mathbf{E}_x(h(\sqrt{t}\xi_r^{(t)})) - \mathbf{E}_x(h(\sqrt{t}\xi_{r+q}^{(t)})) \right|$$

$$+ \sup_{x \in \mathbb{R}} \left| \mathbf{E}_x(g(\xi_r^{(t)})) - \mathbf{E}_x(g(\xi_{r+q}^{(t)})) \right| + \mathbf{E}_y \left(\int_r^{r+q} \psi^{(t)} \left(v_{g,h}^{(t)}(r+q-s, \xi_s^{(t)}) \right) ds \right)$$

$$+ C_\psi \int_0^r \left(\mathbf{E}_y \left(\left(v_{g,h}^{(t)}(r-s, \xi_s^{(t)}) \right)^{\alpha-1} \right) + \mathbf{E}_y \left(\left(v_{g,h}^{(t)}(r+q-s, \xi_s^{(t)}) \right)^{\alpha-1} \right) \right) \hat{F}_q^f(r-s) ds.$$

Combining Lemma 2.5, (2.15) and Lemma 3.1, for any $q \in (0, 1), t > 1$ and $r \in (0, T]$,

$$\hat{F}_q^f(r) \lesssim \frac{1}{\sqrt{tr}} \wedge 1 + \frac{1}{t} + G_h(r, r+q; t) + \sqrt{q}$$

$$\begin{aligned}
 &+ \frac{1}{\sqrt{r+q}} \int_r^{r+q} \frac{1}{(r+q-s)^{(\alpha-1)/2}} ds + \frac{1}{r^{(\alpha-1)/2}} \int_0^r \widehat{F}_q^f(s) ds \\
 &\lesssim \frac{1}{\sqrt{tr}} \wedge 1 + \frac{1}{t} + G_h(r, r+q; t) + \sqrt{q} + \frac{1}{\sqrt{r}} q^{(3-\alpha)/2} + \frac{1}{r^{(\alpha-1)/2}} \int_0^r \widehat{F}_q^f(s) ds.
 \end{aligned}$$

Define $\widehat{G}_q^f(r) := r^{(\alpha-1)/2} \widehat{F}_q^f(r)$. Then there exists a constant $K_2 = K_2(g, h, T, \psi)$ such that for all $q \in (0, 1), t > 1$ and $r \in (0, T]$,

$$\begin{aligned}
 \widehat{G}_q^f(r) &\leq K_2 r^{(\alpha-1)/2} \left(\frac{1}{\sqrt{tr}} \wedge 1 + \frac{1}{t} + G_h(r, r+q; t) + \sqrt{q} + \frac{1}{\sqrt{r}} q^{(3-\alpha)/2} + \frac{1}{r^{(\alpha-1)/2}} \int_0^r \widehat{F}_q^f(s) ds \right) \\
 &=: \widehat{\alpha}_f(r) + K_2 \int_0^r \frac{\widehat{G}_q^f(s)}{s^{(\alpha-1)/2}} ds.
 \end{aligned}$$

Therefore, by Gronwall's inequality, we get that for all $q \in (0, 1), t > 1$ and $r \in (0, T]$,

$$\begin{aligned}
 \widehat{G}_q^f(r) &\leq \widehat{\alpha}_f(r) + K_2 \int_0^r \exp \left\{ K_2 \int_s^r \frac{1}{q^{(\alpha-1)/2}} dq \right\} \frac{\widehat{\alpha}_f(s)}{s^{(\alpha-1)/2}} ds \\
 &\lesssim \widehat{\alpha}_f(r) + \int_0^r \frac{\widehat{\alpha}_f(s)}{s^{(\alpha-1)/2}} ds.
 \end{aligned} \tag{3.9}$$

Using an argument similar to that leading to (3.7) and the fact that $(3 - \alpha)/2 > 1/8$, we get that for all $q \in (0, 1), t > 1$ and $r \in (0, T]$,

$$\begin{aligned}
 \widehat{\alpha}_f(r) &\lesssim r^{(\alpha-1)/2} \left(\frac{1}{t^{1/4} \sqrt{r}} + \frac{\sqrt{T}}{t^{1/4} \sqrt{r}} + G_h(r, r+q; t) + \frac{\sqrt{T}}{\sqrt{r}} q^{1/8} + \frac{1}{\sqrt{r}} q^{1/8} \right) \\
 &\lesssim r^{(\alpha-1)/2} \left(\frac{1}{t^{1/4} \sqrt{r}} + G_h(r, r+q; t) + \frac{1}{\sqrt{r}} q^{1/8} \right).
 \end{aligned} \tag{3.10}$$

Moreover, by the definition of G_h , we know that for all $r \in (0, T], q \in (0, 1)$ and $t > 1$,

$$\begin{aligned}
 G_h(r, r+q; t) &\leq \frac{2\widetilde{\epsilon}_{tr}(h)}{\sqrt{r}} + \ell(h) \left(\frac{q}{\sqrt{r(r+q)}(\sqrt{r} + \sqrt{r+q})} + \frac{1}{\sqrt{r+q}} \left(\sqrt{q} + \frac{\sqrt{q}}{r} \right) \right) \\
 &\lesssim \frac{\widetilde{\epsilon}_{tr}(h)}{\sqrt{r}} + \frac{q^{1/8}}{\sqrt{r^3}} + \frac{1}{\sqrt{r}} \left(1 + \frac{1}{r} \right) q^{1/8} \lesssim \frac{1}{r^{3/2}} \left(\widetilde{\epsilon}_{tr}(h) + q^{1/8} \right).
 \end{aligned} \tag{3.11}$$

Combining (3.10) and (3.11), we see that for all $t > 1, r \in (0, T]$ and $q \in (0, 1)$,

$$\widehat{\alpha}_f(r) \lesssim r^{(\alpha-1)/2} \cdot \frac{1}{r^{3/2}} \left(\frac{1}{t^{1/4}} + q^{1/8} + \widetilde{\epsilon}_{tr}(h) \right). \tag{3.12}$$

Using (3.10), we get that for all $t > 1, r \in (0, T]$ and $q \in (0, 1)$,

$$\begin{aligned}
 \int_0^r \frac{\widehat{\alpha}_f(s)}{s^{(\alpha-1)/2}} ds &\lesssim \int_0^r \frac{1}{t^{1/4} \sqrt{s}} ds + \int_0^r G_h(s, s+q; t) ds + \int_0^r \frac{q^{1/8}}{\sqrt{s}} ds \\
 &\lesssim \frac{1}{t^{1/4}} + \int_0^r G_h(s, s+q; t) ds + q^{1/8}.
 \end{aligned} \tag{3.13}$$

Note that by (3.8),

$$\begin{aligned}
 \int_0^r G_h(s, s+q; t) ds &\leq \int_0^r \frac{2\widetilde{\epsilon}_{ts}(h)}{\sqrt{s}} ds + \int_0^r \left(\frac{1}{\sqrt{s}} - \frac{1}{\sqrt{s+q}} + \frac{1}{\sqrt{s+q}} \left(\sqrt{q} + 1 - e^{-\frac{\sqrt{q}}{s}} \right) \right) ds \\
 &\lesssim \left(\frac{1}{t^{1/4}} + \widetilde{\epsilon}_{tr}(h) \right) + \int_0^r \left(\frac{1}{\sqrt{s}} - \frac{1}{\sqrt{s+q}} + \frac{1}{\sqrt{s+q}} \left(\sqrt{q} + 1 - e^{-\frac{\sqrt{q}}{s}} \right) \right) ds
 \end{aligned} \tag{3.14}$$

and that

$$\begin{aligned}
 &\int_0^r \left(\frac{1}{\sqrt{s}} - \frac{1}{\sqrt{s+q}} + \frac{1}{\sqrt{s+q}} \left(\sqrt{q} + 1 - e^{-\frac{\sqrt{q}}{s}} \right) \right) ds \\
 &= 2(\sqrt{r} - \sqrt{r+q} + \sqrt{q}) + \int_0^r \left(\frac{1}{\sqrt{s+q}} \left(\sqrt{q} + 1 - e^{-\frac{\sqrt{q}}{s}} \right) \right) ds \\
 &\leq 2\sqrt{q} + 2 \int_0^{q^{1/4}} \frac{1}{\sqrt{s+q}} ds + \int_{q^{1/4}}^{r \vee q^{1/4}} \frac{1}{\sqrt{s+q}} \left(\sqrt{q} + \frac{\sqrt{q}}{s} \right) ds \\
 &\leq 2q^{1/8} + 4\sqrt{q+q^{1/4}} + (\sqrt{q} + q^{1/4}) \int_0^T \frac{1}{\sqrt{s}} ds \lesssim q^{1/8}.
 \end{aligned} \tag{3.15}$$

Combining (3.13), (3.14) and (3.15), we get that for all $t > 1$, $q \in (0, 1)$ and $r \in (0, T]$,

$$\int_0^r \frac{\widehat{\alpha}_r(s)}{s^{(\alpha-1)/2}} ds \lesssim \frac{1}{t^{1/4}} + \widetilde{\epsilon}_{\sqrt{tr}}(h) + q^{1/8} \lesssim r^{(\alpha-1)/2} \times \frac{1}{r^{3/2}} \left(\frac{1}{t^{1/4}} + q^{1/8} + \widetilde{\epsilon}_{\sqrt{tr}}(h) \right). \tag{3.16}$$

Now combining (3.9), (3.12) and (3.16), we get (3.5). \square

By (3.3), we see that for any $r > 0$, $h \in \text{DRI}_c^+(\mathbb{R})$ and $g \in B_{Lip}^+(\mathbb{R})$, we have $\sup_{y \in \mathbb{R}, t > 1} v_{g,h}^{(t)}(r, y) < \infty$. Therefore, by a diagonalization argument, for any sequence of positive reals increasing to ∞ , we can find a subsequence $\{t_k : k \in \mathbb{N}\}$ such that $\lim_{k \rightarrow \infty} t_k = \infty$ and that the following limit exists:

$$\lim_{k \rightarrow \infty} v_{g,h}^{(t_k)}(r, y) =: v_{g,h}^X(r, y), \quad \text{for all } r \in \mathbb{Q}_+ := (0, \infty) \cap \mathbb{Q}, y \in \mathbb{Q}. \tag{3.17}$$

Of course, the choice of $\{t_k\}$ may depend on the functions h and g . Using (2.14), taking $t = t_k$ in Proposition 3.2 first and then letting $k \rightarrow \infty$, we see that $v_{g,h}^X(r, y)$ is continuous in $\mathbb{Q}_+ \times \mathbb{Q}$. Now for each $r > 0, y \in \mathbb{R}$, for any sequence $\{(r_m, y_m), m \in \mathbb{N}\} \subset \mathbb{Q}_+ \times \mathbb{Q}$ with $r_m \rightarrow r$ and $y_m \rightarrow y$ as $m \rightarrow \infty$, we see that the sequence of $\{v_{g,h}^X(r_m, y_m), m \in \mathbb{N}\}$ is a Cauchy sequence. Therefore, for each $r > 0$ and $y \in \mathbb{R}$, we can define

$$v_{g,h}^X(r, y) := \lim_{(r_m, y_m) \in \mathbb{Q}_+ \times \mathbb{Q}, (r_m, y_m) \rightarrow (r, y)} v_{g,h}^X(r_m, y_m).$$

Our next result shows that (3.17) also holds for all $r > 0, y \in \mathbb{R}$.

Lemma 3.3. *The limits (3.17) holds for all $r > 0$ and $y \in \mathbb{R}$.*

Proof. Let $(r_m, y_m) \in \mathbb{Q}_+ \times \mathbb{Q}$ be such that $(r_m, y_m) \rightarrow (r, y)$. Without loss of generality, we assume that $2r > r_m > \frac{1}{2}r$ for all m . Then by Proposition 3.2 with $T = 2r$, there exist positive constants $N_1 = N_1(g, h, r, \psi)$ and $N_2 = N_2(g, h, r, \psi)$ such that

$$\begin{aligned} & \left| v_{g,h}^X(r, y) - v_{g,h}^{(t_k)}(r, y) \right| \leq \left| v_{g,h}^X(r, y) - v_{g,h}^X(r_m, y_m) \right| + \left| v_{g,h}^X(r_m, y_m) - v_{g,h}^{(t_k)}(r_m, y_m) \right| \\ & \quad + \left| v_{g,h}^{(t_k)}(r_m, y_m) - v_{g,h}^{(t_k)}(r, y) \right| \\ & \leq \left| v_{g,h}^X(r, y) - v_{g,h}^X(r_m, y_m) \right| + \left| v_{g,h}^X(r_m, y_m) - v_{g,h}^{(t_k)}(r_m, y_m) \right| \\ & \quad + \frac{N_1 2^{3/4}}{r^{3/4}} \left(\widetilde{\epsilon}_{\sqrt{tr/2}}(h) + \sqrt{|r_m - r|} + \frac{1}{t_k^{1/4}} \right) + \frac{N_2 2^{3/2}}{r^{3/2}} \left(\widetilde{\epsilon}_{\sqrt{tr/2}}(h) + |y_m - y|^{1/8} + \frac{1}{t_k^{1/4}} \right). \end{aligned}$$

Letting $k \rightarrow \infty$ in the inequality above and using (3.17), we get

$$\begin{aligned} & \limsup_{k \rightarrow \infty} \left| v_{g,h}^X(r, y) - v_{g,h}^{(t_k)}(r, y) \right| \\ & \leq \left| v_{g,h}^X(r, y) - v_{g,h}^X(r_m, y_m) \right| + \frac{2^{3/4}}{r^{3/4}} N_1 \sqrt{|r_m - r|} + N_2 \frac{2^{3/2} |y_m - y|^{1/8}}{r^{3/2}}. \end{aligned}$$

Letting $m \rightarrow \infty$, we arrive at the desired result. This completes the proof of the lemma. \square

Define

$$\eta_{g,h}(dx) := \ell(h)\delta_0(dx) + g(x)dx.$$

Proposition 3.4. *Any subsequential limit $v_{g,h}^X(r, y)$ of $\{v_{g,h}^{(t)}(r, y)\}$ is equal to the solution $v_{(\emptyset, \eta_{g,h})}^X(r, y)$ of (2.25) with initial trace $(\emptyset, \eta_{g,h})$ whose probabilistic representation is given in Proposition 2.10.*

Proof. By the uniqueness of solutions to (2.25) and Proposition 2.10, we only need to prove that any subsequential limit $v_{g,h}^X(r, y)$ is the solution of (2.25). We divide the proof to two steps.

Step 1: In this step we derive the integral equation for any subsequential limit $v_{g,h}^X(r, y)$. Noticing that $\frac{1}{\alpha-1} \geq 1$, by (2.4), for any $r > 0$ and $y \in \mathbb{R}$, it holds that

$$\begin{aligned} & \limsup_{t \rightarrow \infty} \left| t^{\frac{1}{\alpha-1}} \mathbf{E}_y \left(1 - \exp \left\{ -\frac{1}{t^{\frac{1}{\alpha-1}-\frac{1}{2}}} h(\sqrt{t}\xi_r^{(t)}) - \frac{1}{t^{\frac{1}{\alpha-1}}} g(\xi_r^{(t)}) \right\} \right) - \sqrt{t} \mathbf{E}_y \left(h(\sqrt{t}\xi_r^{(t)}) \right) - \mathbf{E}_y \left(g(\xi_r^{(t)}) \right) \right| \\ & \leq \limsup_{t \rightarrow \infty} \left(\frac{t^{\frac{1}{\alpha-1}}}{t^{\frac{2}{\alpha-1}-1}} \mathbf{E}_y \left(h^2(\sqrt{t}\xi_r^{(t)}) \right) + \frac{t^{\frac{1}{\alpha-1}}}{t^{\frac{2}{\alpha-1}}} \mathbf{E}_y \left(g^2(\xi_r^{(t)}) \right) \right) \\ & \leq \limsup_{t \rightarrow \infty} \left(\|h\|_\infty \times \frac{C(g, h)}{\sqrt{tr}} + \frac{1}{t} \|g^2\|_\infty \right) = 0, \end{aligned} \tag{3.18}$$

where $C(g, h)$ is defined in (2.15). Note that by (2.19),

$$\sqrt{t} \mathbf{E}_y \left(h(\sqrt{t}\xi_r^{(t)}) \right) - \frac{\ell(h)}{\sqrt{r}} \phi \left(\frac{y}{\sqrt{r}} \right) = \frac{1}{\sqrt{r}} \left(\sqrt{tr} \mathbf{E}_{\sqrt{ty}} h(\xi_{tr}) - \ell(h) \phi \left(\frac{\sqrt{ty}}{\sqrt{tr}} \right) \right).$$

Thus by Lemma 2.4,

$$\lim_{t \rightarrow \infty} \sqrt{t} \mathbf{E}_y \left(h(\sqrt{t} \xi_r^{(t)}) \right) = \frac{\ell(h)}{\sqrt{r}} \phi \left(\frac{y}{\sqrt{r}} \right). \tag{3.19}$$

By the central limit theorem we know that $\xi_r^{(t)}$ converges weakly to B_r . Then, combining (3.18) and (3.19), we get that

$$\begin{aligned} & \lim_{t \rightarrow \infty} t^{\frac{1}{a-1}} \mathbf{E}_y \left(1 - \exp \left\{ -\frac{1}{t^{\frac{1}{a-1}-\frac{1}{2}}} h(\sqrt{t} \xi_r^{(t)}) - \frac{1}{t^{\frac{1}{a-1}}} g(\xi_r^{(t)}) \right\} \right) \\ &= \frac{\ell(h)}{\sqrt{r}} \phi \left(\frac{y}{\sqrt{r}} \right) + \mathbf{E}_y (g(B_r)). \end{aligned} \tag{3.20}$$

Now letting $t = t_k$ in Lemma 2.9 first and then $k \rightarrow \infty$, by (3.20), for any $r > 0$ and $y \in \mathbb{R}$,

$$v_{g,h}^X(r, y) = \frac{\ell(h)}{\sqrt{r}} \phi \left(\frac{y}{\sqrt{r}} \right) + \mathbf{E}_y (g(B_r)) - \lim_{k \rightarrow \infty} \mathbf{E}_y \left(\int_0^r \psi^{(t_k)} (v_{g,h}^{(t_k)}(r-s, \xi_s^{(t_k)})) ds \right). \tag{3.21}$$

Combining (2.20) and (3.3), for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $s \in (0, r - \varepsilon)$ and $y \in \mathbb{R}$,

$$(1 - \varepsilon) \varphi (v_{g,h}^{(t_k)}(r-s, y)) \leq \psi^{(t_k)} (v_{g,h}^{(t_k)}(r-s, y)) \leq (1 + \varepsilon) \varphi (v_{g,h}^{(t_k)}(r-s, y)), \tag{3.22}$$

where φ is the function defined in (1.12). Thus, for $k > N$, we have

$$\begin{aligned} \mathbf{E}_y \left(\int_0^r \psi^{(t_k)} (v_{g,h}^{(t_k)}(r-s, \xi_s^{(t_k)})) ds \right) &\geq \mathbf{E}_y \left(\int_0^{r-\varepsilon} \psi^{(t_k)} (v_{g,h}^{(t_k)}(r-s, \xi_s^{(t_k)})) ds \right) \\ &\geq (1 - \varepsilon) \mathbf{E}_y \left(\int_0^{r-\varepsilon} \varphi (v_{g,h}^{(t_k)}(r-s, \xi_s^{(t_k)})) ds \right). \end{aligned} \tag{3.23}$$

Similarly, combining Lemma 2.8, (3.3) and (3.22), we get that for $k > N$,

$$\begin{aligned} & \mathbf{E}_y \left(\int_0^r \psi^{(t_k)} (v_{g,h}^{(t_k)}(r-s, \xi_s^{(t_k)})) ds \right) \\ &\leq (1 + \varepsilon) \mathbf{E}_y \left(\int_0^{r-\varepsilon} \varphi (v_{g,h}^{(t_k)}(r-s, \xi_s^{(t_k)})) ds \right) + C_\psi \int_{r-\varepsilon}^r \left(\frac{C}{\sqrt{r-s}} \right)^\alpha ds \\ &= (1 + \varepsilon) \mathbf{E}_y \left(\int_0^{r-\varepsilon} \varphi (v_{g,h}^{(t_k)}(r-s, \xi_s^{(t_k)})) ds \right) + f(\varepsilon), \end{aligned} \tag{3.24}$$

where C is a positive constant, and $f(\varepsilon)$ is a function of ε satisfying $\lim_{\varepsilon \rightarrow 0} f(\varepsilon) = 0$. We claim that for any $r > \varepsilon > 0$ and $y \in \mathbb{R}$,

$$\lim_{k \rightarrow \infty} \mathbf{E}_y \left(\int_0^{r-\varepsilon} \varphi (v_{g,h}^{(t_k)}(r-s, \xi_s^{(t_k)})) ds \right) = \mathbf{E}_y \left(\int_0^{r-\varepsilon} \varphi (v_{g,h}^X(r-s, B_s)) ds \right). \tag{3.25}$$

To prove (3.25), fix two large constants R and $T > r$, since $v_{g,h}^X(r, y)$ is continuous in $(r, y) \in [\varepsilon, T] \times [-R, R]$, for any $\gamma_0 \in (0, 1)$, there exist $L \in \mathbb{N}$ and $r_0 = \varepsilon < \dots < r_L = T$, $y_0 := -R < \dots < y_L = R$ such that $\max_{i \in \{1, \dots, L\}} |r_i - r_{i-1}| < \gamma_0$, $\max_{i \in \{1, \dots, L\}} |y_i - y_{i-1}| < \gamma_0$ and that

$$\max_{i,j \in \{1, \dots, L\}} \max_{r \in [r_{i-1}, r_i], y \in [y_{j-1}, y_j]} |v_{g,h}^X(r_i, y_j) - v_{g,h}^X(r, y)| < \gamma_0. \tag{3.26}$$

Now we take N_* sufficiently large so that when $k > N_*$,

$$\max_{r \in \{r_0, \dots, r_L\}, y \in \{y_0, \dots, y_L\}} |v_{g,h}^{(t_k)}(r, y) - v_{g,h}^X(r, y)| < \gamma_0. \tag{3.27}$$

Combining Proposition 3.2, (3.26) and (3.27), we get that, if $k > N_*$, then for any $r \in [\varepsilon, T]$, $y \in [-R, R]$, suppose that $r_{i_0-1} \leq r \leq r_{i_0}$ and $y_{j_0-1} \leq y \leq y_{j_0}$ for some $i_0, j_0 \in \{1, \dots, L\}$,

$$\begin{aligned} & |v_{g,h}^{(t_k)}(r, y) - v_{g,h}^X(r, y)| < \gamma_0 + |v_{g,h}^{(t_k)}(r, y) - v_{g,h}^X(r_{i_0}, y_{j_0})| \\ &< 2\gamma_0 + |v_{g,h}^{(t_k)}(r, y) - v_{g,h}^{(t_k)}(r_{i_0}, y_{j_0})| \\ &\leq 2\gamma_0 + \frac{N_1}{\varepsilon^{3/4}} \left(\tilde{c}_{\sqrt{t_k \varepsilon}}(h) + \sqrt{|y - y_{j_0}|} + \frac{1}{t_k^{1/4}} \right) + \frac{N_2}{\varepsilon^{3/2}} \left(\tilde{c}_{\sqrt{t_k \varepsilon}}(h) + |r - r_{i_0}|^{1/8} + \frac{1}{t_k^{1/4}} \right) \\ &\leq 2\gamma_0^{1/8} + \left(\frac{N_1}{\varepsilon^{3/4}} + \frac{N_2}{\varepsilon^{3/2}} \right) \left(\tilde{c}_{\sqrt{t_k \varepsilon}}(h) + \gamma_0^{1/8} + \frac{1}{t_k^{1/4}} \right). \end{aligned}$$

Note that $\lim_{t \rightarrow \infty} \tilde{c}_{\sqrt{t \varepsilon}}(h) = 0$ by (2.14). Therefore, for any $\gamma_0 \in (0, 1)$, there exists a constant $C' = C'(\varepsilon, T, R)$ such that when k is large enough,

$$|v_{g,h}^{(t_k)}(r, y) - v_{g,h}^X(r, y)| \leq C' \gamma_0^{1/8},$$

for any $r \in [\varepsilon, T]$ and $y \in [-R, R]$. Therefore, combining (3.3) and the fact that $|\varphi(u) - \varphi(v)| \leq C_{\varphi, \varepsilon} |u - v|$ for all $u, v \in [0, \frac{C}{\sqrt{\varepsilon}}]$, we obtain that for any $\gamma \in (0, 1)$, $R > 1$ and $T > r$,

$$\begin{aligned} & \limsup_{k \rightarrow \infty} \left| \mathbf{E}_y \left(\int_0^{r-\varepsilon} \varphi \left(v_{g,h}^{(t_k)}(r-s, \xi_s^{(t_k)}) \right) ds \right) - \mathbf{E}_y \left(\int_0^{r-\varepsilon} \varphi \left(v_{g,h}^X(r-s, \xi_s^{(t_k)}) \right) ds \right) \right| \\ & \leq 2 \limsup_{k \rightarrow \infty} \int_0^{r-\varepsilon} \varphi \left(\frac{C}{\sqrt{r-s}} \right) \mathbf{P}_y \left(|\xi_s^{(t_k)}| > R \right) ds \\ & \quad + C_{\varphi, \varepsilon} \limsup_{k \rightarrow \infty} \mathbf{E}_y \left(\int_0^{r-\varepsilon} \left| v_{g,h}^{(t_k)}(r-s, \xi_s^{(t_k)}) - v_{g,h}^X(r-s, \xi_s^{(t_k)}) \right| 1_{\{\xi_s^{(t_k)} \in [-R, R]\}} ds \right) \\ & \leq 2\varphi \left(\frac{C}{\sqrt{\varepsilon}} \right) \int_0^{r-\varepsilon} \frac{s+y^2}{R^2} ds + C_{\varphi, \varepsilon} C' \gamma_0^{1/8} (r-\varepsilon)^{\gamma_0 \downarrow 0, R \uparrow \infty} \rightarrow 0. \end{aligned}$$

By the functional central limit theorem we know that $(\xi_s^{(t)})_{0 < s \leq r-\varepsilon}$ converges weakly to $(B_s)_{0 < s \leq r-\varepsilon}$, thus (3.25) is valid. Plugging (3.25) into (3.23) and (3.24), we conclude that for any $\varepsilon > 0$,

$$\begin{aligned} & (1-\varepsilon) \mathbf{E}_y \left(\int_0^{r-\varepsilon} \varphi \left(v_{g,h}^X(r-s, B_s) \right) ds \right) \\ & \leq \liminf_{k \rightarrow \infty} \mathbf{E}_y \left(\int_0^r \psi^{(t_k)} \left(v_{g,h}^{(t_k)}(r-s, \xi_s^{(t_k)}) \right) ds \right) \leq \limsup_{k \rightarrow \infty} \mathbf{E}_y \left(\int_0^r \psi^{(t_k)} \left(v_{g,h}^{(t_k)}(r-s, \xi_s^{(t_k)}) \right) ds \right) \\ & \leq (1+\varepsilon) \mathbf{E}_y \left(\int_0^{r-\varepsilon} \varphi \left(v_{g,h}^X(r-s, B_s) \right) ds \right) + f(\varepsilon). \end{aligned}$$

Letting $\varepsilon \downarrow 0$ in the inequality above and plugging the resulting inequality into (3.21), we get that for any $r > 0$ and $y \in \mathbb{R}$,

$$v_{g,h}^X(r, y) = \frac{\ell(h)}{\sqrt{r}} \phi \left(\frac{y}{\sqrt{r}} \right) + \mathbf{E}_y(g(B_r)) - \mathbf{E}_y \left(\int_0^r \varphi \left(v_{g,h}^X(r-s, B_s) \right) ds \right). \tag{3.28}$$

Step 2: In this step we show that (3.28) is equivalent to (2.25) with initial trace $(\emptyset, \eta_{g,h})$. Combining (3.28) and the Markov property, we see that for $w \in (0, r)$,

$$\begin{aligned} & v_{g,h}^X(r, y) + \mathbf{E}_y \left(\int_0^w \varphi \left(v_{g,h}^X(r-s, B_s) \right) ds \right) \\ & = \frac{\ell(h)}{\sqrt{r}} \phi \left(\frac{y}{\sqrt{r}} \right) + \mathbf{E}_y(g(B_r)) - \mathbf{E}_y \left(\int_w^r \varphi \left(v_{g,h}^X(r-s, B_s) \right) ds \right) \\ & = \frac{\ell(h)}{\sqrt{r}} \phi \left(\frac{y}{\sqrt{r}} \right) + \mathbf{E}_y \left(\mathbf{E}_{B_w}(g(B_{r-w})) - \mathbf{E}_{B_w} \left(\int_0^{r-w} \varphi \left(v_{g,h}^X(r-s, B_s) \right) ds \right) \right) \\ & = \frac{\ell(h)}{\sqrt{r}} \phi \left(\frac{y}{\sqrt{r}} \right) - \mathbf{E}_y \left(\frac{\ell(h)}{\sqrt{r-w}} \phi \left(\frac{B_w}{\sqrt{r-w}} \right) \right) + \mathbf{E}_y \left(v_{g,h}^X(r-w, B_w) \right). \end{aligned} \tag{3.29}$$

Routine computations yield that

$$\begin{aligned} & \frac{\ell(h)}{\sqrt{r}} \phi \left(\frac{y}{\sqrt{r}} \right) - \mathbf{E}_y \left(\frac{\ell(h)}{\sqrt{r-w}} \phi \left(\frac{B_w}{\sqrt{r-w}} \right) \right) \\ & = \frac{\ell(h)}{\sqrt{2\pi}} \left(\frac{1}{\sqrt{r}} e^{-y^2/(2r)} - \frac{1}{\sqrt{2\pi w(r-w)}} \int \exp \left\{ -\frac{rz^2}{2w(r-w)} + \frac{zy}{w} - \frac{y^2}{2w} \right\} dz \right) \\ & = \frac{\ell(h)}{\sqrt{2\pi}} \left(\frac{1}{\sqrt{r}} e^{-y^2/(2r)} - \frac{1}{\sqrt{2\pi w(r-w)}} \int \exp \left\{ -\frac{r}{2w(r-w)} \left(z - \frac{r-w}{r} y \right)^2 - \frac{y^2}{2r} \right\} dz \right) \\ & = 0. \end{aligned} \tag{3.30}$$

Therefore, combining (3.29) and (3.30), we conclude that

$$v_{g,h}^X(r, y) + \mathbf{E}_y \left(\int_0^w \varphi \left(v_{g,h}^X(r-s, B_s) \right) ds \right) = \mathbf{E}_y \left(v_{g,h}^X(r-w, B_w) \right). \tag{3.31}$$

For any fixed $w > 0$, set $u(r, y) := v_{g,h}^X(r+w, y)$, then we see from (3.31) that u solves (1.13) with $f = v_{g,h}^X(w, \cdot)$. Now it suffices to check the boundary condition. By (3.28), for any $j \in C_c^+(\mathbb{R})$ and any $r > 0$,

$$\begin{aligned} & \int j(y) v_{g,h}^X(r, y) dy \\ & = \ell(h) \int j(y) \frac{1}{\sqrt{r}} \phi \left(\frac{y}{\sqrt{r}} \right) + \int j(y) \mathbf{E}_y(g(B_r)) ds - \int j(y) \mathbf{E}_y \left(\int_0^r \varphi \left(v_{g,h}^X(r-s, B_s) \right) ds \right) dy \end{aligned}$$

$$= \ell(h)\mathbf{E}_0(j(B_r)) + \int j(y)\mathbf{E}_y(g(B_r)) \, ds - \int j(y)\mathbf{E}_y\left(\int_0^r \varphi\left(v_{g,h}^X(r-s, B_s)\right) \, ds\right) \, dy.$$

Since $\lim_{r \downarrow 0} \mathbf{E}_0(j(B_r)) = j(0)$ and $\lim_{r \downarrow 0} \int j(y)\mathbf{E}_y(g(B_r)) \, ds = \int j(y)g(y) \, dy$ by the dominated convergence theorem, to prove the desired result, we only need to prove that

$$\lim_{r \downarrow 0} \int j(y)\mathbf{E}_y\left(\int_0^r \varphi\left(v_{g,h}^X(r-s, B_s)\right) \, ds\right) \, dy = 0.$$

Combining (3.3) and the definition of φ , we see that

$$\begin{aligned} & \int j(y)\mathbf{E}_y\left(\int_0^r \varphi\left(v_{g,h}^X(r-s, B_s)\right) \, ds\right) \, dy \\ & \lesssim \int j(y)\mathbf{E}_y\left[\int_0^r \frac{1}{(\sqrt{r-s})^{\alpha-1}} v_{g,h}^X(r-s, B_s) \, ds\right] \, dy \\ & = \int_0^r \frac{1}{(r-s)^{(\alpha-1)/2}} \int j(y)\mathbf{E}_y\left(v_{g,h}^X(r-s, B_s)\right) \, dy \, ds \\ & \leq \int_0^r \frac{1}{(r-s)^{(\alpha-1)/2}} \int j(y)\mathbf{E}_y\left(\frac{\ell(h)}{\sqrt{r-s}} \phi\left(\frac{B_s}{\sqrt{r-s}}\right) + \mathbf{E}_{B_s}(g(B_{r-s}))\right) \, dy \, ds, \end{aligned} \tag{3.32}$$

where in the last inequality we used (3.28). Combining (3.30) and (3.32), we get

$$\begin{aligned} & \int j(y)\mathbf{E}_y\left(\int_0^r \varphi\left(v_{g,h}^X(r-s, B_s)\right) \, ds\right) \, dy \\ & \lesssim \int_0^r \frac{1}{(r-s)^{(\alpha-1)/2}} \int j(y)\left(\frac{\ell(h)}{\sqrt{r}} \phi\left(\frac{y}{\sqrt{r}}\right) + \|g\|_\infty\right) \, dy \, ds \\ & = \int_0^r \frac{1}{s^{(\alpha-1)/2}} \, ds \left(\ell(h)\mathbf{E}_0(j(B_r)) + \|g\|_\infty \int j(y) \, dy\right) \\ & \lesssim (\ell(h)\|j\|_\infty + \ell(j)\|g\|_\infty) \int_0^r \frac{1}{s^{(\alpha-1)/2}} \, ds \xrightarrow{r \downarrow 0} 0, \end{aligned}$$

which implies the desired result. \square

Proof of Theorem 1.2. Theorem 1.2 follows directly from Proposition 3.4. \square

Now we are going to prove Theorem 1.4. Before the proof, we need to prove an upper bound for maximal position $M_t := \max_{u \in N(t)} X_u(t)$ and the minimal position $M_t^- := \min_{u \in N(t)} X_u(t)$ of all the particle alive at time t with the convention that $M_t = -\infty$ and $M_t^- = \infty$ when $N(t) = \emptyset$. In the next lemma, we will need (H4') to control the overshoot of Lévy process.

Lemma 3.5. Assume (H1), (H2), (H3) and (H4') hold. For any $q, \delta > 0$, there exist constants $C(q), T(\delta) \in (0, \infty)$ such that for $t > T(\delta)$,

$$t^{\frac{1}{\alpha-1}} \mathbb{P}_0\left(M_{t\delta} > q\sqrt{t}\right) \leq C(q)\delta \quad \text{and} \quad t^{\frac{1}{\alpha-1}} \mathbb{P}_0\left(M_{t\delta}^- < -q\sqrt{t}\right) \leq C(q)\delta.$$

Proof. We only prove the first inequality, the proof of the second one is similar. Set

$$Q^{(t)}(r, x) := t^{\frac{1}{\alpha-1}} \mathbb{P}_{\sqrt{tx}}\left(M_{tr} > 0\right) = \lim_{\theta \uparrow \infty} t^{\frac{1}{\alpha-1}} \left(1 - \mathbb{E}_{\sqrt{tx}}\left(\exp\{-\theta Z_{tr}((0, \infty))\}\right)\right).$$

Then $t^{\frac{1}{\alpha-1}} \mathbb{P}_0\left(M_{t\delta} > q\sqrt{t}\right) = Q^{(t)}(\delta, -q)$, and we only need to prove that there exist constants $C(q), T(\delta) \in (0, \infty)$ such that for $t > T(\delta)$,

$$Q^{(t)}(\delta, -q) \leq C(q)\delta. \tag{3.33}$$

By Lemma 2.9 (with $h = \theta 1_{(0, \infty)}, g = 0$ first and then $\theta \uparrow \infty$), we see that $Q^{(t)}(r, x)$ solves

$$Q^{(t)}(r, x) = t^{\frac{1}{\alpha-1}} \mathbf{P}_x\left(\xi_r^{(t)} > 0\right) - \mathbf{E}_x\left(\int_0^r \psi^{(t)}\left(Q^{(t)}(r-s, \xi_s^{(t)})\right) \, ds\right).$$

By the Markov property, for any $w < r$, the above equation can also be rewritten by

$$Q^{(t)}(r, x) = \mathbf{E}_x\left(Q^{(t)}(r-w, \xi_w^{(t)})\right) - \mathbf{E}_x\left(\int_0^w \psi^{(t)}\left(Q^{(t)}(r-s, \xi_s^{(t)})\right) \, ds\right).$$

It follows from the Feynman-Kac formula that for any $0 < w < r$,

$$Q^{(t)}(r, x) = \mathbf{E}_y\left(\exp\left\{-\int_0^w K^{(t)}\left(Q^{(t)}(r-s, \xi_s^{(t)})\right) \, ds\right\} Q^{(t)}(r-w, \xi_w^{(t)})\right),$$

where

$$K^{(t)}(v) := \frac{1}{v} \psi^{(t)}(v)$$

and $\psi^{(t)}(v)$ is defined in (2.17). Also by the Markov property of $\xi^{(t)}$, we see that for all $y \in \mathbb{R}$ and $w \in [0, r]$, it holds that

$$\begin{aligned} Y_w &:= \exp \left\{ - \int_0^w K^{(t)}(Q^{(t)}(r-s, \xi_s^{(t)})) ds \right\} Q^{(t)}(r-w, \xi_w^{(t)}) \\ &= \mathbf{E}_y \left(\exp \left\{ - \int_0^r K^{(t)}(Q^{(t)}(r-s, \xi_s^{(t)})) ds \right\} Q^{(t)}(0, \xi_r^{(t)}) \Big| \xi_s^{(t)} : s \leq w \right). \end{aligned}$$

Therefore, $\{(Y_w)_{w \in [0,r]}, \mathbf{P}_y\}$ is a non-negative martingale, which implies that for any stopping time S ,

$$\begin{aligned} Q^{(t)}(r, x) &= \mathbf{E}_x(Y_{w \wedge S}) \\ &= \mathbf{E}_x \left(\exp \left\{ - \int_0^{w \wedge S} K^{(t)}(Q^{(t)}(r-s, \xi_s^{(t)})) ds \right\} Q^{(t)}(r-w \wedge S, \xi_{w \wedge S}^{(t)}) \right). \end{aligned}$$

In particular, set $S = \tau_{-q/2}^{(t,+)} := \inf \{ r > 0 : \xi_r^{(t)} \geq -q/2 \}$ and $r = w = \delta$, we see that

$$\begin{aligned} Q^{(t)}(\delta, -q) &\leq \mathbf{E}_{-q} \left(Q^{(t)}(\delta - \delta \wedge \tau_{-q/2}^{(t,+)}, \xi_{\delta \wedge \tau_{-q/2}^{(t,+)} }^{(t)}) \right) \\ &= \mathbf{E}_{-q} \left(Q^{(t)} \left(\delta - \tau_{-q/2}^{(t,+)}, \xi_{\tau_{-q/2}^{(t,+)} }^{(t)} \right) 1_{\{\tau_{-q/2}^{(t,+)} \leq \delta\}} \right), \end{aligned} \tag{3.34}$$

where in the last equality we used the fact that on the event $\{\delta < \tau_{-q/2}^{(t,+)}\} = \{\sup_{s \leq \delta} \xi_s^{(t)} < -q/2\}$, it holds that

$$Q^{(t)}(\delta - \delta \wedge \tau_{-q/2}^{(t,+)}, \xi_{\delta \wedge \tau_{-q/2}^{(t,+)} }^{(t)}) = Q^{(t)}(0, \xi_\delta^{(t)}) = t^{\frac{1}{a-1}} 1_{\{\xi_\delta^{(t)} > 0\}} = 0.$$

Note that $Q^{(t)}(\delta, -q) \leq t^{\frac{1}{a-1}}$. Note also that, by (1.11), for any $z < -q/4$ and $r > 0$,

$$Q^{(t)}(r, z) \leq t^{\frac{1}{a-1}} \mathbb{P}_0(M > -z\sqrt{t}) \leq t^{\frac{1}{a-1}} \mathbb{P}_0(M > q\sqrt{t}/4) \lesssim q^{-\frac{2}{a-1}}.$$

Comparing $\xi_{\tau_{-q/2}^{(t,+)} }^{(t)}$ with $-q/4$, using (3.34) and the two facts above, we get that

$$\begin{aligned} Q^{(t)}(\delta, -q) &\lesssim t^{\frac{1}{a-1}} \mathbf{P}_{-q} \left(\xi_{\tau_{-q/2}^{(t,+)} }^{(t)} > -\frac{q}{4} \right) + q^{-\frac{2}{a-1}} \mathbf{P}_{-q} \left(\delta \geq \tau_{-q/2}^{(t,+)} \right) \\ &= t^{\frac{1}{a-1}} \mathbf{P}_{-q\sqrt{t}/2} \left(\xi_{\tau_0^+} > \frac{q\sqrt{t}}{4} \right) + q^{-\frac{2}{a-1}} \mathbf{P}_0 \left(\sup_{s \leq t\delta} \xi_s \geq q\sqrt{t}/2 \right). \end{aligned}$$

Now combining the above inequality, Lemma 2.6 and Doob's maximal inequality, we get that

$$\begin{aligned} Q^{(t)}(\delta, -q) &\lesssim \frac{t^{\frac{1}{a-1}}}{(q\sqrt{t})^{r_0-2}} \sup_{x>0} \mathbf{E}_{-x} \left(\xi_{\tau_0^+}^{r_0-2} \right) + \frac{1}{q^{\frac{2}{a-1}+2t}} \mathbf{E}_0(|\xi_{t\delta}|^2) \\ &\lesssim \frac{1}{q^{r_0-2} t^{\frac{r_0-2}{2} - \frac{1}{a-1}}} + \frac{\delta}{q^{\frac{2a}{a-1}}}. \end{aligned}$$

Since $\frac{r_0-2}{2} - \frac{1}{a-1} > 0$ under (H4'), letting $T(\delta)$ be sufficiently large so that $t^{\frac{r_0-2}{2} - \frac{1}{a-1}} > \delta^{-1}$ for all $t > T(\delta)$, we get the desired (3.33). \square

Proof of Theorem 1.4. By Theorem 1.2, for any $\theta > 0$, we have the following lower bound for the lim inf:

$$\begin{aligned} \liminf_{t \rightarrow \infty} t^{\frac{1}{a-1}} \mathbb{P}_{\sqrt{t}y} (Z_t(A) > 0) &\geq \lim_{t \rightarrow \infty} t^{\frac{1}{a-1}} \left(1 - \mathbb{E}_{\sqrt{t}y} \left(\exp \left\{ -\frac{\theta}{t^{\frac{1}{a-1}-\frac{1}{2}}} Z_t(A) \right\} \right) \right) \\ &= -\log \mathbb{E}_{\delta_y} (\exp \{-\theta \ell(A) Y_1(0)\}) \xrightarrow{\theta \uparrow \infty} -\log \mathbb{P}_{\delta_y} (Y_1(0) = 0). \end{aligned} \tag{3.35}$$

Now we prove the lim sup is no larger than the right-hand side above. By the branching property, for $\kappa > 0$, it holds that

$$\mathbb{P}_{\sqrt{t}y} (Z_{(1+\kappa)t}(A) > 0) = \mathbb{E}_{\sqrt{t}y} \left(1 - \exp \left\{ \int \log \mathbb{P}_a(Z_{t\kappa}(A) = 0) Z_t(da) \right\} \right).$$

Noticing that for all $a \in \mathbb{R}$,

$$\mathbb{P}_a(Z_{t\kappa}(A) = 0) \geq \mathbb{P}_a(Z_{t\kappa}(\mathbb{R}) = 0) = \mathbb{P}_0(Z_{t\kappa}(\mathbb{R}) = 0) \xrightarrow{t \rightarrow \infty} 1.$$

Using the fact that $\log x \sim x - 1$ as $x \rightarrow 1$, we see that there exists $N(\kappa)$ such that as $t > N(\kappa)$,

$$\mathbb{P}_{\sqrt{t}y} (Z_{(1+\kappa)t}(A) > 0) \leq \mathbb{E}_{\sqrt{t}y} \left(1 - \exp \left\{ -\frac{1}{2} \int \mathbb{P}_a(Z_{t\kappa}(A) > 0) Z_t(da) \right\} \right). \tag{3.36}$$

Now we fix a small $\varepsilon > 0$. Suppose that t is large enough such that $A \subset [-\varepsilon\sqrt{t}, \varepsilon\sqrt{t}]$. For $a < -2\varepsilon\sqrt{t}$, by Lemma 3.5, when t is large enough, we have

$$\mathbb{P}_a(Z_{t\kappa}(A) > 0) \leq \mathbb{P}_{-2\varepsilon\sqrt{t}}(M_{t\kappa} > -\varepsilon\sqrt{t}) \leq \frac{C(\varepsilon)\kappa}{t^{\frac{1}{\alpha-1}}}. \tag{3.37}$$

Similarly, for $a > 2\varepsilon\sqrt{t}$, when t is sufficient large, it holds that

$$\mathbb{P}_a(Z_{t\kappa}(A) > 0) \leq \mathbb{P}_{2\varepsilon\sqrt{t}}(M_{t\kappa}^- \leq \varepsilon\sqrt{t}) \leq \frac{C(\varepsilon)\kappa}{t^{\frac{1}{\alpha-1}}}. \tag{3.38}$$

When $|a| \leq 2\varepsilon\sqrt{t}$, by (1.10), we have

$$\mathbb{P}_a(Z_{t\kappa}(A) > 0) \leq \mathbb{P}_0 (Z_{t\kappa}(\mathbb{R}) > 0) \leq \frac{C_*}{(t\kappa)^{\frac{1}{\alpha-1}}} \tag{3.39}$$

for some constant $C_* \in (0, \infty)$. Combining (3.36), (3.37), (3.38), (3.39) and the inequality $1 - e^{-|x|-|y|} \leq (1 - e^{-|x|}) + |y|$, we obtain that

$$\begin{aligned} t^{\frac{1}{\alpha-1}} \mathbb{P}_{\sqrt{t}y} (Z_{(1+\kappa)t}(A) > 0) &\leq t^{\frac{1}{\alpha-1}} \mathbb{E}_{\sqrt{t}y} \left(1 - \exp \left\{ -\frac{C_*}{2\kappa^{\frac{1}{\alpha-1}} t^{\frac{1}{\alpha-1}}} Z_t([-2\varepsilon\sqrt{t}, 2\varepsilon\sqrt{t}]) \right\} \right) \\ &\quad + \frac{C(\varepsilon)\kappa}{2} \mathbb{E}_{\sqrt{t}y} (Z_t((-\infty, -2\varepsilon\sqrt{t}) \cup (2\varepsilon\sqrt{t}, \infty))) \\ &\leq t^{\frac{1}{\alpha-1}} \mathbb{E}_{\sqrt{t}y} \left(1 - \exp \left\{ -\frac{C_*}{2\kappa^{\frac{1}{\alpha-1}} t^{\frac{1}{\alpha-1}}} Z_t([-2\varepsilon\sqrt{t}, 2\varepsilon\sqrt{t}]) \right\} \right) + \frac{C(\varepsilon)\kappa}{2}, \end{aligned} \tag{3.40}$$

where in the last inequality we used the fact that $\mathbb{E}_{\sqrt{t}y} (Z_t(\mathbb{R})) = 1$. This is because, thanks to the branching property and the criticality of our branching Lévy process, $(Z_t(\mathbb{R}))_{t \geq 0}$ is a martingale under \mathbb{P}_x with $Z_0(\mathbb{R}) = 1$ \mathbb{P}_x -a.s. for any $x \in \mathbb{R}$. Now define a function

$$f(x) := \frac{C_*}{2\kappa^{\frac{1}{\alpha-1}}} \left(\left(2 - \frac{1}{2\varepsilon} |x| \right)_+ \wedge 1 \right), x \in \mathbb{R},$$

then we see f is a bounded continuous function with support equal to $[-4\varepsilon, 4\varepsilon]$ and $f = \frac{C_*}{2\kappa^{\frac{1}{\alpha-1}}}$ for $x \in [-2\varepsilon, 2\varepsilon]$. Plugging this observation into (3.40), we get that for large t ,

$$\begin{aligned} &t^{\frac{1}{\alpha-1}} \mathbb{P}_{\sqrt{t}y} (Z_{(1+\kappa)t}(A) > 0) \\ &\leq t^{\frac{1}{\alpha-1}} \mathbb{E}_{\sqrt{t}y} \left(1 - \exp \left\{ -\frac{1}{t^{\frac{1}{\alpha-1}}} \int f \left(\frac{a}{\sqrt{t}} \right) Z_t(da) \right\} \right) + \frac{C(\varepsilon)\kappa}{2} \\ &= t^{\frac{1}{\alpha-1}} \mathbb{E}_{\sqrt{t}y} \left(1 - \exp \left\{ -\int f(a) Z_1^{(t)}(da) \right\} \right) + \frac{C(\varepsilon)\kappa}{2}. \end{aligned} \tag{3.41}$$

Letting $t \rightarrow \infty$ in (3.41), using Theorem 1.2 with $h = 0$ and $g = f$, we get that

$$\begin{aligned} \limsup_{t \rightarrow \infty} t^{\frac{1}{\alpha-1}} \mathbb{P}_{\sqrt{t}y} (Z_t(A) > 0) &= (1 + \kappa)^{\frac{1}{\alpha-1}} \limsup_{t \rightarrow \infty} t^{\frac{1}{\alpha-1}} \mathbb{P}_{\sqrt{t}\sqrt{1+\kappa}y} (Z_{(1+\kappa)t}(A) > 0) \\ &\leq (1 + \kappa)^{\frac{1}{\alpha-1}} \lim_{t \rightarrow \infty} t^{\frac{1}{\alpha-1}} \mathbb{E}_{\sqrt{t}\sqrt{1+\kappa}y} \left(1 - \exp \left\{ -\int f(a) Z_1^{(t)}(da) \right\} \right) + (1 + \kappa)^{\frac{1}{\alpha-1}} \frac{C(\varepsilon)\kappa}{2} \\ &= -(1 + \kappa)^{\frac{1}{\alpha-1}} \log \mathbb{E}_{\delta_{\sqrt{1+\kappa}y}} \left(\exp \left\{ -\int f(a) X_1(da) \right\} \right) + \frac{C(\varepsilon)(1 + \kappa)^{\frac{1}{\alpha-1}}}{2} \kappa \\ &\leq -(1 + \kappa)^{\frac{1}{\alpha-1}} \log \mathbb{P}_{\delta_{\sqrt{1+\kappa}y}} (X_1([-4\varepsilon, 4\varepsilon]) = 0) + \frac{C(\varepsilon)(1 + \kappa)^{\frac{1}{\alpha-1}}}{2} \kappa, \end{aligned}$$

where in the last inequality we used the fact that f is supported in $[-4\varepsilon, 4\varepsilon]$. Letting $\kappa \rightarrow 0$ in the above inequality, we see that

$$\limsup_{t \rightarrow \infty} t^{\frac{1}{\alpha-1}} \mathbb{P}_{\sqrt{t}y} (Z_t(A) > 0) \leq -\log \mathbb{P}_{\delta_y} (X_1([-4\varepsilon, 4\varepsilon]) = 0) = -\log \mathbb{P}_{\delta_y} (Y_1(x) = 0, \forall |x| \leq 4\varepsilon).$$

Taking $\varepsilon \rightarrow 0$ and using Remark 2.12, we conclude that

$$\limsup_{t \rightarrow \infty} t^{\frac{1}{\alpha-1}} \mathbb{P}_{\sqrt{t}y} (Z_t(A) > 0) \leq -\log \mathbb{P}_{\delta_y} (Y_1(0) = 0). \tag{3.42}$$

Combining (3.35) and (3.42), we complete the proof of Theorem 1.4. \square

Proof of Theorem 1.6. (i) For any $f \in C_c^+(\mathbb{R})$, it holds that

$$\begin{aligned} & \mathbb{E}_{\sqrt{t}y} \left(\exp \left\{ -\frac{1}{t^{\frac{1}{\alpha-1}-\frac{1}{2}}} \int f(x)Z_t(dx) \right\} \mid Z_t(A) > 0 \right) \\ &= 1 - \frac{1}{\mathbb{P}_{\sqrt{t}y}(Z_t(A) > 0)} \mathbb{E}_{\sqrt{t}y} \left(\left(1 - \exp \left\{ -\frac{1}{t^{\frac{1}{\alpha-1}-\frac{1}{2}}} \int f(x)Z_t(dx) \right\} \right) 1_{\{Z_t(A) > 0\}} \right). \end{aligned} \tag{3.43}$$

Let $B = (a, b)$ be a bounded interval such that $\text{supp}(f) \subset B$ and $A \subset B$. Then by Theorem 1.4, we see that

$$\begin{aligned} & \left| \frac{1}{\mathbb{P}_{\sqrt{t}y}(Z_t(A) > 0)} \mathbb{E}_{\sqrt{t}y} \left(\left(1 - \exp \left\{ -\frac{1}{t^{\frac{1}{\alpha-1}-\frac{1}{2}}} \int f(x)Z_t(dx) \right\} \right) (1_{\{Z_t(B) > 0\}} - 1_{\{Z_t(A) > 0\}}) \right) \right| \\ & \leq \frac{1}{\mathbb{P}_{\sqrt{t}y}(Z_t(A) > 0)} \left(\mathbb{P}_{\sqrt{t}y}(Z_t(B) > 0) - \mathbb{P}_{\sqrt{t}y}(Z_t(A) > 0) \right) \xrightarrow{t \rightarrow \infty} 0. \end{aligned} \tag{3.44}$$

Further, combining Theorem 1.2 and Theorem 1.4, we get that

$$\begin{aligned} & \lim_{t \rightarrow \infty} \frac{1}{\mathbb{P}_{\sqrt{t}y}(Z_t(A) > 0)} \mathbb{E}_{\sqrt{t}y} \left(\left(1 - \exp \left\{ -\frac{1}{t^{\frac{1}{\alpha-1}-\frac{1}{2}}} \int f(x)Z_t(dx) \right\} \right) 1_{\{Z_t(B) > 0\}} \right) \\ &= \lim_{t \rightarrow \infty} \frac{1}{\mathbb{P}_{\sqrt{t}y}(Z_t(A) > 0)} \mathbb{E}_{\sqrt{t}y} \left(1 - \exp \left\{ -\frac{1}{t^{\frac{1}{\alpha-1}-\frac{1}{2}}} \int f(x)Z_t(dx) \right\} \right) \\ &= \frac{1}{\log \mathbb{P}_{\delta_y}(Y_1(0) = 0)} \log \mathbb{E}_{\delta_y} \left(\exp \left\{ -Y_1(0) \int f(x)dx \right\} \right). \end{aligned} \tag{3.45}$$

Therefore, combining (3.43), (3.44) and (3.45) we see that

$$\begin{aligned} & \lim_{t \rightarrow \infty} \mathbb{E}_{\sqrt{t}y} \left(\exp \left\{ -\frac{1}{t^{\frac{1}{\alpha-1}-\frac{1}{2}}} \int f(x)Z_t(dx) \right\} \mid Z_t(A) > 0 \right) \\ &= 1 - \frac{1}{\log \mathbb{P}_{\delta_y}(Y_1(0) = 0)} \log \mathbb{E}_{\delta_y} \left(\exp \left\{ -Y_1(0) \int f(x)dx \right\} \right). \end{aligned} \tag{3.46}$$

For any $\theta, \varepsilon > 0$, taking $f = h + \frac{\theta}{2\varepsilon} 1_{[-\varepsilon, \varepsilon]}$ in (1.14) and letting $\varepsilon \rightarrow 0$, by Lemma 1.1, we have

$$\begin{aligned} & \mathbb{N}_y(1 - e^{-\theta Y_1(0) - w_1(h)}) = \lim_{\varepsilon \rightarrow 0} \mathbb{N}_y \left(1 - \exp \left\{ -\frac{\theta}{2\varepsilon} w_1([-\varepsilon, \varepsilon]) - w_1(h) \right\} \right) \\ &= -\lim_{\varepsilon \rightarrow 0} \log \mathbb{E}_{\delta_y} \left(\exp \left\{ -\frac{\theta}{2\varepsilon} X_1([-\varepsilon, \varepsilon]) - X_1(h) \right\} \right) \\ &= -\log \mathbb{E}_{\delta_y} \left(\exp \left\{ -\theta Y_1(0) - X_1(h) \right\} \right). \end{aligned} \tag{3.47}$$

Letting $h = 0$ and $\theta \uparrow \infty$, we also see that

$$\mathbb{N}_y(Y_1(0) > 0) = -\log \mathbb{E}_{\delta_y}(Y_1(0) = 0). \tag{3.48}$$

Combining (3.46), (3.47) and (3.48), we have that

$$\begin{aligned} & \lim_{t \rightarrow \infty} \mathbb{E}_{\sqrt{t}y} \left(\exp \left\{ -\frac{1}{t^{\frac{1}{\alpha-1}-\frac{1}{2}}} \int f(x)Z_t(dx) \right\} \mid Z_t(A) > 0 \right) \\ &= 1 - \frac{1}{\mathbb{N}_y(Y_1(0) > 0)} \mathbb{N}_y \left(1 - \exp \left\{ -Y_1(0) \int f(x)dx \right\} \right) \\ &= 1 - \frac{1}{\mathbb{N}_y(Y_1(0) > 0)} \mathbb{N}_y \left(\left(1 - \exp \left\{ -Y_1(0) \int f(x)dx \right\} \right) 1_{\{Y_1(0) > 0\}} \right) \\ &= \mathbb{N}_y \left(\exp \left\{ -Y_1(0) \int f(x)dx \right\} \mid Y_1(0) > 0 \right), \end{aligned}$$

which implies (i).

(ii) To prove the convergence in distribution in the weak topology, it suffices to prove that for any $g \in B_{Lip}^+(\mathbb{R})$ (for example, see [21, Lemma 3.4]),

$$\begin{aligned} & \lim_{t \rightarrow \infty} \mathbb{E}_{\sqrt{t}y} \left(\exp \left\{ -\frac{1}{t^{\frac{1}{\alpha-1}}} \int g \left(\frac{y}{\sqrt{t}} \right) Z_t(dy) \right\} \mid Z_t(A) > 0 \right) \\ &= \mathbb{N}_y \left(\exp \left\{ -w_1(g) \right\} \mid Y_1(0) > 0 \right). \end{aligned}$$

Note that

$$\mathbb{E}_{\sqrt{t}y} \left(\exp \left\{ -\frac{1}{t^{\frac{1}{\alpha-1}}} \int g \left(\frac{y}{\sqrt{t}} \right) Z_t(dy) \right\} \mid Z_t(A) > 0 \right)$$

$$= \frac{1}{\mathbb{P}_{\sqrt{t}y}(Z_t(A) > 0)} \mathbb{E}_{\sqrt{t}y} \left(\exp \left\{ -\frac{1}{t^{\frac{1}{a-1}}} \int g \left(\frac{y}{\sqrt{t}} \right) Z_t(dy) \right\} 1_{\{Z_t(A) > 0\}} \right). \tag{3.49}$$

Since $1_{\{|x|>0\}} \geq 1 - e^{-a|x|}$ for $a \geq 0$, by Theorems 1.2 and 1.4, for any $\theta \in (0, \infty)$, we have that

$$\begin{aligned} & \limsup_{t \rightarrow \infty} t^{\frac{1}{a-1}} \left| \mathbb{E}_{\sqrt{t}y} \left(\exp \left\{ -\frac{1}{t^{\frac{1}{a-1}}} \int g \left(\frac{y}{\sqrt{t}} \right) Z_t(dy) \right\} 1_{\{Z_t(A) > 0\}} \right) \right. \\ & \quad \left. - \mathbb{E}_{\sqrt{t}y} \left(\exp \left\{ -\frac{1}{t^{\frac{1}{a-1}}} \int g \left(\frac{y}{\sqrt{t}} \right) Z_t(dy) \right\} \left(1 - \exp \left\{ -\frac{\theta}{t^{\frac{1}{a-1}-\frac{1}{2}}} Z_t(A) \right\} \right) \right) \right| \\ & \leq \lim_{t \rightarrow \infty} t^{\frac{1}{a-1}} \left| \mathbb{P}_{\sqrt{t}y}(Z_t(A) > 0) - \mathbb{E}_{\sqrt{t}y} \left(1 - \exp \left\{ -\frac{\theta}{t^{\frac{1}{a-1}-\frac{1}{2}}} Z_t(A) \right\} \right) \right| \\ & = \left| -\log \mathbb{P}_{\delta_y}(Y_1(0) = 0) + \log \mathbb{E}_{\delta_y}(\exp \{ -\theta \ell(A) Y_1(0) \}) \right| =: G(\theta). \end{aligned}$$

Therefore, combining the above inequality and Theorem 1.4, we conclude that for any $\theta \in (0, \infty)$,

$$\begin{aligned} & \limsup_{t \rightarrow \infty} t^{\frac{1}{a-1}} \mathbb{E}_{\sqrt{t}y} \left(\exp \left\{ -\frac{1}{t^{\frac{1}{a-1}}} \int g \left(\frac{y}{\sqrt{t}} \right) Z_t(dy) \right\} 1_{\{Z_t(A) > 0\}} \right) \\ & \leq G(\theta) - \log \mathbb{E}_{\delta_y}(\exp \{ -\theta \ell(A) Y_1(0) - X_1(g) \}) + \log \mathbb{E}_{\delta_y}(\exp \{ -X_1(g) \}) \\ & \xrightarrow{\theta \uparrow \infty} -\log \mathbb{E}_{\delta_y}(\exp \{ -X_1(g) \} 1_{\{Y_1(0)=0\}}) + \log \mathbb{E}_{\delta_y}(\exp \{ -X_1(g) \}). \end{aligned} \tag{3.50}$$

Similarly, we also have that

$$\begin{aligned} & \liminf_{t \rightarrow \infty} t^{\frac{1}{a-1}} \mathbb{E}_{\sqrt{t}y} \left(\exp \left\{ -\frac{1}{t^{\frac{1}{a-1}}} \int g \left(\frac{y}{\sqrt{t}} \right) Z_t(dy) \right\} 1_{\{Z_t(A) > 0\}} \right) \\ & \geq -G(\theta) - \log \mathbb{E}_{\delta_y}(\exp \{ -\theta \ell(A) Y_1(0) - X_1(g) \}) + \log \mathbb{E}_{\delta_y}(\exp \{ -X_1(g) \}) \\ & \xrightarrow{\theta \uparrow \infty} -\log \mathbb{E}_{\delta_y}(\exp \{ -X_1(g) \} 1_{\{Y_1(0)=0\}}) + \log \mathbb{E}_{\delta_y}(\exp \{ -X_1(g) \}). \end{aligned} \tag{3.51}$$

Combining Theorem 1.4, (3.49) (3.50) and (3.51), we conclude that

$$\begin{aligned} & \lim_{t \rightarrow \infty} \mathbb{E}_{\sqrt{t}y} \left(\exp \left\{ -\frac{1}{t^{\frac{1}{a-1}}} \int g \left(\frac{y}{\sqrt{t}} \right) Z_t(dy) \right\} \middle| Z_t(A) > 0 \right) \\ & = \frac{1}{-\log \mathbb{P}_{\delta_y}(Y_1(0) = 0)} \left(-\log \mathbb{E}_{\delta_y}(\exp \{ -X_1(g) \} 1_{\{Y_1(0)=0\}}) + \log \mathbb{E}_{\delta_y}(\exp \{ -X_1(g) \}) \right). \end{aligned} \tag{3.52}$$

Combining (3.47), (3.48) and (3.52), we get that

$$\begin{aligned} & \lim_{t \rightarrow \infty} \mathbb{E}_{\sqrt{t}y} \left(\exp \left\{ -\frac{1}{t^{\frac{1}{a-1}}} \int g \left(\frac{y}{\sqrt{t}} \right) Z_t(dy) \right\} \middle| Z_t(A) > 0 \right) \\ & = \frac{1}{\mathbb{N}_y(Y_1(0) > 0)} \left(\mathbb{N}_y(1 - e^{-w_1(g)} 1_{\{Y_1(0)=0\}}) - \mathbb{N}_y(1 - e^{-w_1(g)}) \right) \\ & = \frac{1}{\mathbb{N}_y(Y_1(0) > 0)} \mathbb{N}_y(e^{-w_1(g)} 1_{\{Y_1(0)>0\}}) = \mathbb{N}_y(e^{-w_1(g)} | Y_1(0) > 0), \end{aligned}$$

which implies the desired result. \square

4. Proof of Lemma 1.1

Proof of Lemma 1.1. Suppose that Q_t is the transition semigroup of the super Brownian motion, i.e.

$$Q_t(v_1, dv_2) := \mathbb{P}_{v_1}(X_t \in dv_2).$$

Let Q_t° be the restriction of Q_t on $\mathcal{M}_F(\mathbb{R}) \setminus \{\mathbf{0}\}$. By [20, A.41 or (8.46)], for any $y \in \mathbb{R}$, $0 < r_1 < \dots < r_m < \infty$ and $v_1, \dots, v_m \in \mathcal{M}_F(\mathbb{R}) \setminus \{\mathbf{0}\}$, we have

$$\mathbb{N}_y(w_{r_1} \in dv_1, \dots, w_{r_m} \in dv_m) = \mathbb{N}_y(w_{r_1} \in dv_1) Q_{r_2-r_1}^\circ(v_1, dv_2) \cdots Q_{r_m-r_{m-1}}^\circ(v_{m-1}, dv_m).$$

In particular, for any $s < t$,

$$\begin{aligned} \mathbb{N}_y(w_s \in \mathcal{M}_F(\mathbb{R}) \setminus \{\mathbf{0}\}, w_t \in \mathcal{A}^c) &= \int_{\mathcal{M}_F(\mathbb{R}) \setminus \{\mathbf{0}\}} \mathbb{N}_y(w_r \in dv_1) Q_{t-r}^\circ(v_1, \mathcal{A}^c) \\ &= \int_{\mathcal{M}_F(\mathbb{R}) \setminus \{\mathbf{0}\}} \mathbb{N}_y(w_r \in dv_1) \mathbb{P}_{v_1}(X_{t-r} \in \mathcal{A}^c), \end{aligned}$$

where in the last equality we used the fact that $\mathbf{0} \notin \mathcal{A}^c$. Since $\mathbb{P}_{\nu_1}(X_{t-r} \in \mathcal{A}) = 1$ for all $\nu_1 \in \mathcal{M}_F(\mathbb{R})$ and $t > r$, we obtain that

$$\mathbb{N}_y(w_r \in \mathcal{M}_F(\mathbb{R}) \setminus \{\mathbf{0}\}, w_t \in \mathcal{A}^c) = 0. \quad (4.1)$$

Moreover, $w_t \in \mathcal{A}^c$ implies that $w_t \neq \mathbf{0}$, therefore, it must hold that $w_r \in \mathcal{M}_F(\mathbb{R}) \setminus \{\mathbf{0}\}$. Therefore, by (4.1), we get that

$$\mathbb{N}_y(w_r \in \mathcal{M}_F(\mathbb{R}) \setminus \{\mathbf{0}\}, w_t \in \mathcal{A}^c) = \mathbb{N}_y(w_t \in \mathcal{A}^c) = 0,$$

which implies the desired result. \square

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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