# Asymptotic behaviors of subcritical branching killed Lévy processes

Yan-Xia Ren [a], Renming Song [b], Yaping Zhu [c,*]

[a] *LMAM School of Mathematical Sciences & Center for Statistical Science, Peking University, Beijing, 100871, PR China*
[b] *Department of Mathematics, University of Illinois, Urbana, IL, 61801, USA*
[c] *Department of Mathematics, Shanghai University of Finance and Economics, Shanghai, 200433, PR China*

## ARTICLE INFO

## ABSTRACT

In this paper, we investigate the asymptotic behaviors of the survival probability and maximal displacement of a subcritical branching killed Lévy process $X$ in $\mathbb{R}$. Let $\zeta$ denote the extinction time, $M_t$ be the maximal position of all the particles alive at time $t$, and $M := \sup_{t \geq 0} M_t$ be the all-time maximum. Under the assumption that the offspring distribution satisfies the $L \log L$ condition and some conditions on the spatial motion, we find the decay rate of the survival probability $\mathbb{P}_x(\zeta > t)$ and the tail behavior of $M_t$ as $t \to \infty$. As a consequence, we establish a Yaglom-type theorem. We also find the asymptotic behavior of $\mathbb{P}_x(M > y)$ as $y \to \infty$.

## 1. Introduction

### 1.1. Background and motivation

A branching Lévy process on $\mathbb{R}$ is defined as follows: at time 0, there is a particle at $x \in \mathbb{R}$ and it moves according to a Lévy process $(\xi_t, \mathbf{P}_x)$ on $\mathbb{R}$. After an exponential time with parameter $\beta > 0$, independent of the spatial motion, this particle dies and is replaced by $k$ offspring with probability $p_k$, $k \geq 0$. The offspring move independently according to the same Lévy process starting from the death position of their parent. This procedure goes on. Let $N_t$ be the set of particles alive at time $t$ and for each $u \in N_t$, we denote by $X_u(t)$ the position of $u$ at time $t$. Also, for any $u \in N_t$ and $s \leq t$, we use $X_u(s)$ to denote the position of $u$ or its ancestor at time $s$. Then the point process $Z = (Z_t)_{t \geq 0}$ defined by

$$Z_t := \sum_{u \in N_t} \delta_{X_u(t)}$$

is called a branching Lévy process. We shall denote by $\mathbb{P}_x$ the law of this process when the initial particle starts from $x$ and use $\mathbb{E}_x$ to denote the corresponding expectation. Let

$$\widetilde{\zeta} := \inf\{t > 0 : Z_t(\mathbb{R}) = 0\}$$

---

be the extinction time of $Z$. Note that $\widetilde{\zeta}$ is equal in law to that of the extinction time of a continuous-time Galton-Watson process with the same branching mechanism as the branching Lévy process. Let $m := \sum_{k=0}^{\infty} kp_k$ be the mean number of offspring. It is well-known that $Z$ will become extinct in finite time with probability 1 if and only if $m < 1$ (subcritical) or $m = 1$ and $p_1 \neq 1$ (critical). Moreover, the process $Z$ survives with positive probability when $m > 1$ (supercritical).

The focus of this paper is on the asymptotic behaviors of a branching killed Lévy process, in which particles are killed upon entering the negative half-line. The point process $Z^0 = (Z_t^0)_{t \geq 0}$ defined by

$$Z_t^0 := \sum_{u \in N_t} 1_{\{\inf_{s \leq t} X_u(s) > 0\}} \delta_{X_u(t)}$$

is called a branching killed Lévy process. For any $t \geq 0$, let

$$M_t := \sup_{u \in N_t, \inf_{s \leq t} X_u(s) > 0} X_u(t)$$

be the maximal position of all the particles alive at time $t$ in the process $Z^0$. We define the all-time maximum position and the extinction time of $Z^0$ by

$$M := \sup_{t \geq 0} M_t, \qquad \zeta := \inf \left\{ t > 0 : Z_t^0((0, \infty)) = 0 \right\}.$$

In the critical case, i.e., when $m = 1$ and $p_1 \neq 1$, the asymptotic behaviors of the tails of the extinction time and the maximal displacement of $Z^0$ were established in [1] under the assumption that the offspring distribution belongs to the domain of attraction of an $\alpha$-stable distribution, $\alpha \in (1, 2]$, and some moment assumptions on the spatial motion. It was also shown in [1] that the scaling limit under $\mathbb{P}_{\sqrt{t}y}(\cdot | \zeta > t)$ can be represented in terms of a super killed Brownian motion. In the subcritical case, i.e., $m \in (0, 1)$, under the assumption $\sum_{k=1}^{\infty} k(\log k) p_k < \infty$, the asymptotic behaviors of the survival probability and the all time maximal position of branching killed Brownian motion with drift were established in [2] recently.

The asymptotic behavior of branching Lévy processes have been studied earlier. In the critical case, i.e. $m = 1$ and $p_1 \neq 1$, Sawyer and Fleischman [3] investigated the tail behavior of the all time maximal position of branching Brownian motion under the assumption that the offspring distribution has finite third moment. For a critical branching random walk with spatial motion having finite $(4 + \varepsilon)$th moment, the tail behavior of the all time maximum was obtained by Lalley and Shao [4]. Hou et al. [5] studied the asymptotic behavior of the all time maximum of critical branching Lévy processes with offspring distribution belonging to the domain of attraction of an $\alpha$-stable distribution with $\alpha \in (1, 2]$, under some assumptions on the spatial motion. In the subcritical case, Profeta [6] gave the asymptotic behavior of the all time maximal position under the assumption that the offspring distribution has finite third moment. For related results about subcritical branching random walks, we refer the reader to [7].

The purpose of this paper is to extend the results of [2] to subcritical branching killed Lévy processes. This extension is quite challenging since properties of Brownian motion were used crucially in [2]. Fluctuation theory of Lévy processes will play an important role in this paper. Another important tool is the conditioned limit theorem in Theorem 3.1 below.

## 1.2. Main results

Before we state our main results, we introduce some notation and some basic results on Lévy processes. We always assume that the offspring distribution is subcritical, i.e., $m \in (0, 1)$. Let $\alpha := \beta(1 - m)$ and let $f$ be the generating function of the offspring distribution, i.e. $f(s) = \sum_{k=0}^{\infty} p_k s^k$, $s \in [0, 1]$. Define

$$\Phi(u) := \beta(f(1 - u) - (1 - u)) =: (\alpha + \varphi(u))u, \quad u \in [0, 1], \tag{1.1}$$

where $\varphi(u) = \frac{\Phi(u) - \alpha u}{u}$ for $u \in (0, 1]$ and $\varphi(0) = \Phi'(0+) - \alpha = 0$. According to [2, Lemma 2.7], $\varphi(\cdot)$ is increasing on $[0, 1]$ and under the condition

$$\sum_{k=1}^{\infty} k(\log k) p_k < \infty, \tag{1.2}$$

it holds that

$$\int_0^{\infty} \varphi(e^{-ct}) dt < \infty, \quad \text{for any } c > 0. \tag{1.3}$$

Moreover, it is well-known (see Theorem 2.4 in [8, p.121]) that

$$\lim_{t \to \infty} e^{\alpha t} \mathbb{P}_0(\widetilde{\zeta} > t) = C_{sub} \in (0, \infty) \tag{1.4}$$

holds if and only if (1.2) holds. For any $t > 0$, define

$$g(t) := \mathbb{P}_0(\widetilde{\zeta} > t).$$

It is well-known that $g(t)$ satisfies the equation

$$\frac{\mathrm{d}}{\mathrm{d}t} g(t) = -\Phi(g(t)) = -(\alpha + \varphi(g(t)))g(t),$$

thus

$$e^{\alpha t} g(t) = \exp\left\{ -\int_0^t \varphi(g(s)) \mathrm{d}s \right\}. \tag{1.5}$$

It follows from (1.4) that

$$C_{sub} = \exp\left\{ -\int_0^\infty \varphi(g(s)) \mathrm{d}s \right\}. \tag{1.6}$$

Therefore, (1.2) is equivalent to

$$\int_0^\infty \varphi(g(s)) \mathrm{d}s < \infty.$$

In this paper, we always assume that $\xi = ((\xi_t)_{t \geq 0}, (\mathbf{P}_x)_{x \in \mathbb{R}})$ is a Lévy process on $\mathbb{R}$ with

$$-\log \mathbf{E}_x\left(e^{i\theta(\xi_1 - \xi_0)}\right) = ia\theta + \frac{1}{2}\eta^2\theta^2 + \int_{-\infty}^\infty (1 - e^{i\theta x} + i\theta x 1_{\{|x|<1\}}) \Pi(\mathrm{d}x), \quad \theta \in \mathbb{R},$$

where $\mathbf{E}_x$ stands for the expectation with respect to $\mathbf{P}_x$, $a \in \mathbb{R}$, $\eta \geq 0$ and the Lévy measure $\Pi$ satisfies $\int_{\mathbb{R}} (1 \wedge x^2) \Pi(\mathrm{d}x) < +\infty$. For any $z \in \mathbb{R}$, define

$$\tau_z^+ := \inf\{t > 0 : \xi_t \geq z\} \quad \text{and} \quad \tau_z^- := \inf\{t > 0 : \xi_t \leq z\}.$$

Define the function

$$R(x) := x - \mathbf{E}_x\left(\xi_{\tau_0^-}\right) = -\mathbf{E}_0\left(\xi_{\tau_{-x}^-}\right), \quad x \geq 0. \tag{1.7}$$

It follows from [1, Lemma 2.8] that if $\mathbf{E}_0(\xi_1) = 0$ and $\mathbf{E}_0(\xi_1^2) \in (0, \infty)$, then $\mathbf{E}_x|\xi_{\tau_0^-}| < \infty$ and $R(x)$ satisfies the following:

(1) $R(x) \geq x$ and $R(x)$ is non-decreasing in $x$;
(2) there exists a constant $c > 0$ such that $R(x) \leq c(1 + x)$ and

$$\lim_{x \to \infty} \frac{R(x)}{x} = 1 - \lim_{x \to \infty} \frac{\mathbf{E}_x(\xi_{\tau_0^-})}{x} = 1;$$

(3) $\left( R(\xi_s) 1_{\{\tau_0^- > s\}} \right)_{s \geq 0}$ is a $\mathbf{P}_x$-martingale for any $x > 0$.

In the case $\xi$ is a Brownian motion with drift, it is obvious that

$$R(x) = x, \quad x > 0. \tag{1.8}$$

We remark that the function $R$, defined in (1.7) under the conditions $\mathbf{E}_0(\xi_1) = 0$ and $\mathbf{E}_0(\xi_1^2) \in (0, \infty)$, is related to the function $h$ defined in [9, (2.3)]. Let $\xi^0$ be the process $\xi$ killed upon exiting $(0, \infty)$. According to [10, Theorem 4], if $\xi$ further satisfies the absolute continuity condition, 0 is regular for $(-\infty, 0)$ and $\mathbf{P}_x(\tau_0^+ < \infty) = 1$ for all $x < 0$, then any non-negative harmonic function of $\xi^0$ must be a linear combination of the harmonic function in [10, Theorem 1] and the invariant function in [10, Theorem 2]. Since any invariant function is harmonic and the harmonic function in [10, Theorem 1] is not invariant (see the paragraph above [10, Theorem 2]), we know that, under the conditions mentioned in the previous sentence, all the invariant functions of the process $\xi^0$ are multiples of each other. Property (3) above says that $R$ is an invariant function of $\xi^0$, while the second sentence in [9, Lemma 1] says that the function $h$ defined [9, (2.3)] is also an invariant function of $\xi^0$. Thus $R$ and $h$ must be multiples of each other. This relation is probably true under more general conditions, but we will not pursue this here.

In some results, we will assume that $\xi$ satisfies one or both of the following conditions:

**(H1)** There exists $\delta \in (0, 1)$ such that $\mathbf{E}_x\left(|\xi_1|^{2+\delta}\right) < \infty$.
**(H2)** The law of $\xi_1$ is non-lattice, i.e., $\mathbf{P}_x\left(\xi_1 \in h\mathbb{Z} + a\right) \neq 1, \forall h > 0, a \in [0, h)$.

**Remark 1.** Condition **(H2)** will be assumed in the case $\mathbf{E}_0(\xi_1) < 0$. In this case, we rely on the conditioned limit theorem for random walks established by [11], which requires the non-lattice condition.

In the case $\mathbf{E}_0(\xi_1) < 0$, we will perform an Esscher transform on the Lévy process. For this, we assume that
**(H3)** The Laplace exponent $\Psi(\lambda) := \log \mathbf{E}_0\left(e^{\lambda \xi_1}\right)$ is finite for all $\lambda \in (\Lambda_1, \Lambda_2)$ with $\Lambda_1 \in [-\infty, 0]$ and $\Lambda_2 \in (0, \infty]$. Moreover, there exists a unique $\lambda_* \in (0, \Lambda_2)$ such that $\Psi'(\lambda_*) = 0$.

Note that $\Psi(\lambda)$ is finite if and only if $\int_{\{|x| \geq 1\}} e^{\lambda x} \Pi(\mathrm{d}x) < \infty$ and that for any $\lambda \in (\Lambda_1, \Lambda_2)$,

$$\Psi(\lambda) = a\lambda + \frac{\eta^2}{2}\lambda^2 + \int_{\mathbb{R}} \left(e^{\lambda x} - 1 - \lambda x 1_{\{|x|<1\}}\right) \Pi(\mathrm{d}x).$$

Note also that $\Psi$ is convex in $(\Lambda_1, \Lambda_2)$.

**Remark 2.** If $\xi = ((\xi_t)_{t \geq 0}, (\mathbf{P}_x)_{x \in \mathbb{R}})$ is a spectrally negative Lévy process, then $\xi$ has finite Laplace exponent in $(0, \infty)$. If $\xi$ is a spectrally negative Lévy process satisfying $\mathbf{E}_0(\xi_1) < 0$ and $\Psi(\infty) = \infty$, then $\Psi$ admits a unique minimum at a $\lambda_* > 0$ and $\Psi(\lambda_*) < 0$, $\Psi'(\lambda_*) = 0$ and $\Psi''(\lambda_*) > 0$. So in this case **(H3)** is automatically satisfied.

For any $c \in (\Lambda_1, \Lambda_2)$ and $x \in \mathbb{R}$, since $\{e^{c(\xi_t - x) - \Psi(c)t} : t \geq 0\}$ is a $\mathbf{P}_x$-martingale, we can define the change of measure

$$\frac{d\mathbf{P}_x^c}{d\mathbf{P}_x}\Big|_{\mathcal{F}_t} = e^{c(\xi_t - x) - \Psi(c)t}, \tag{1.9}$$

where $\mathcal{F}_t := \sigma\{\xi_s : s \leq t\}$, $t \geq 0$. According to [12, Theorem 3.9], $\xi^{(c)} = ((\xi_t)_{t \geq 0}, (\mathbf{P}_x^c)_{x \in \mathbb{R}})$ is also a Lévy process and its Laplace exponent $\Psi_c(\lambda)$ is given by $\Psi_c(\lambda) := \Psi(\lambda + c) - \Psi(c)$. We will use $\mathbf{E}_x^c$ to denote expectation with respect to $\mathbf{P}_x^c$.

Recall that $\int_{\{|x| \geq 1\}} e^{\lambda x} \Pi(dx) < \infty$ for any $\lambda \in (\Lambda_1, \Lambda_2)$. According to [12, Theorem 3.9], the Lévy measure of $\xi^{(c)}$ is given by $e^{cx}\Pi(dx)$. Combining the two facts above, we get that, if $\xi$ has finite $p$-th moment with $p \geq 1$, then for any $c \in (\Lambda_1, \Lambda_2)$, $\xi^{(c)}$ also has finite $p$-th moment and so $\xi^{(c)}$ satisfies **(H1)**. It is also easy to see that $\xi^{(c)}$ is non-lattice if and only if $\xi$ is non-lattice. We note that, by [12, Theorem 3.9], (i) if $\xi$ is a spectrally negative Lévy process with Laplace exponent $\Psi$, then $\xi^{(c)}$ is a spectrally negative Lévy process with Laplace exponent $\Psi_c(\lambda)$ given by $\Psi_c(\lambda) := \Psi(\lambda + c) - \Psi(c)$; and (ii) if $\xi$ is a Brownian motion with drift, $\xi^{(c)}$ is also a Brownian motion with drift.

When $\mathbf{E}_0(\xi_1) < 0$ and **(H3)** holds, we take $c = \lambda_*$ and define the change of measure

$$\frac{d\mathbf{P}_x^{\lambda_*}}{d\mathbf{P}_x}\Big|_{\mathcal{F}_t} = e^{\lambda_*(\xi_t - x) - \Psi(\lambda_*)t}. \tag{1.10}$$

Then $\xi^{(\lambda_*)}$ is a Lévy process and its Laplace exponent is given by $\Psi_{\lambda_*}(\lambda) := \Psi(\lambda + \lambda_*) - \Psi(\lambda_*)$. It is easy to see that $\Psi'_{\lambda_*}(0+) = \Psi'(\lambda_*) = 0$. Let $\mathbf{E}_x^{\lambda_*}$ be the expectation with respect to $\mathbf{P}_x^{\lambda_*}$. If $\xi$ satisfies **(H1)**, then since $\mathbf{E}_0^{\lambda_*}(\xi_1) = \Psi'_{\lambda_*}(0+) = 0$, by [1, Lemma 2.8], we have $\mathbf{E}_x^{\lambda_*}|\xi_{\tau_0^-}| < \infty$. Define

$$R^*(x) = x - \mathbf{E}_x^{\lambda_*}\left(\xi_{\tau_0^-}\right), \quad x \geq 0. \tag{1.11}$$

Define the dual process of $\xi$ by:

$$\hat{\xi}_s := -\xi_s, \quad s \geq 0.$$

For any $z \in \mathbb{R}$, we define $\hat{\tau}_z^- := \inf\{s > 0 : \hat{\xi}_s \leq z\}$ and

$$\hat{R}^*(x) := x - \mathbf{E}_x^{\lambda_*}\left(\hat{\xi}_{\hat{\tau}_0^-}\right), \quad x \geq 0. \tag{1.12}$$

Denote $\mathbb{R}_+ = [0, \infty)$. Let $\sigma^2 := \mathbf{E}_0(\xi_1^2)$. Our first main result is on the large-time asymptotic behavior of the survival probability.

**Theorem 1.1.** *Assume* (1.2) *holds and $\xi$ is a Lévy process satisfying* **(H1)**. *Let $x > 0$.*

*(1) If $\mathbf{E}_0(\xi_1) = 0$, then*

$$\lim_{t \to \infty} \sqrt{t} e^{\alpha t} \mathbb{P}_x(\zeta > t) = \frac{2C_{sub}R(x)}{\sqrt{2\pi\sigma^2}},$$

*where $C_{sub}$ is defined in* (1.6) *and $R(x)$ in* (1.7).
*(2) If $\mathbf{E}_0(\xi_1) > 0$, then*

$$\lim_{t \to \infty} e^{\alpha t} \mathbb{P}_x(\zeta > t) = q_x C_{sub},$$

*where $q_x := \mathbf{P}_x\left(\tau_0^- = \infty\right) > 0$.*
*(3) If $\mathbf{E}_0(\xi_1) < 0$ and $\xi$ satisfies* **(H2)** *and* **(H3)**, *then*

$$\lim_{t \to \infty} t^{3/2} e^{(\alpha - \Psi(\lambda_*))t} \mathbb{P}_x(\zeta > t) = \frac{2C_0 R^*(x) e^{\lambda_* x}}{\sqrt{2\pi\Psi''(\lambda_*)^3}},$$

*where $C_0 := \lim_{N \to \infty} e^{(\alpha - \Psi(\lambda_*))N} \int_{\mathbb{R}_+} \mathbb{P}_z(\zeta > N) e^{-\lambda_* z} \hat{R}^*(z) dz \in (0, \infty)$, $R^*$ is defined in* (1.11) *and $\hat{R}^*$ in* (1.12).

**Remark 3.** [2, Theorem 1.1] investigates the asymptotic behavior of the survival probability of a branching killed Brownian motion with drift $-\rho$. The first two statements of [2, Theorem 1.1] are as follows.

(1) if $\rho = 0$, then $\lim_{t \to \infty} \sqrt{t} e^{\alpha t} \mathbf{P}_x(\zeta > t) = \sqrt{\frac{2}{\pi}} C_{sub} x$.

(2) If $\rho < 0$, then $\lim_{t \to \infty} e^{\alpha t} \mathbf{P}_x(\zeta > t) = (1 - e^{2\rho x})$.

Combining Theorem 1.1 (1) and (2) with (1.8), we immediately recover the first two conclusions of [2, Theorem 1.1]. Furthermore, when $\xi$ is a standard Brownian motion with drift $-\rho$, we have $\Psi(\lambda) = -\rho\lambda + \frac{1}{2}\lambda^2$ and $\lambda_* = \rho$. When $\rho > 0$, a straightforward calculation yields that

$$\lim_{t \to \infty} t^{3/2} e^{(\alpha + \frac{\rho^2}{2})t} \mathbb{P}_x(\zeta > t) = \frac{2C_0 x e^{\rho x}}{\sqrt{2\pi}}.$$

This result is consistent with [2, Theorem 1.1, (iii)], where $C_0(\rho) = C_0$.

Our second main result is on the asymptotic behavior of the tail probability of $M_t$.

**Theorem 1.2.** *Assume* (1.2) *holds and $\xi$ is a Lévy process satisfying* **(H1)**. *Let $x > 0$.*

*(1) If $\mathbf{E}_0(\xi_1) = 0$, then for any $y \geq 0$, we have*

$$\lim_{t \to \infty} \sqrt{t} e^{\alpha t} \mathbb{P}_x\left(M_t > \sqrt{t} y\right) = \frac{2C_{sub} R(x)}{\sqrt{2\pi\sigma^2}} e^{-\frac{y^2}{2\sigma^2}}.$$

*(2) If $\mathbf{E}_0(\xi_1) > 0$, then for any $y \in \mathbb{R}$, we have*

$$\lim_{t \to \infty} e^{\alpha t} \mathbb{P}_x\left(M_t > \sqrt{t} y + \mathbf{E}_0(\xi_1) t\right) = \frac{q_x C_{sub}}{\sqrt{2\pi}} \int_{\frac{y}{\sigma}}^{\infty} e^{-\frac{z^2}{2}} \, dz.$$

*(3) If $\mathbf{E}_0(\xi_1) < 0$ and $\xi$ satisfies* **(H2)** *and* **(H3)**, *then for any $y \geq 0$, we have*

$$\lim_{t \to \infty} t^{3/2} e^{(\alpha - \Psi(\lambda_*))t} \mathbb{P}_x\left(M_t > y\right) = \frac{2C_1(y) R^*(x) e^{\lambda_* x}}{\sqrt{2\pi \Psi''(\lambda_*)^3}},$$

*where $C_1(y) := \lim_{N \to \infty} e^{(\alpha - \Psi(\lambda_*))N} \int_{\mathbb{R}_+} \mathbb{P}_z(M_N > y) e^{-\lambda_* z} \widehat{R}^*(z) dz \in (0, \infty)$.*

Note that $\mathbb{P}_z(M_N > 0) = \mathbb{P}_z(\zeta > N)$ for $z > 0$, thus $C_0$ in Theorem 1.1 and $C_1(0)$ in Theorem 1.2 are the same. Combining the result above with (1.8) and (1.13), we immediately recover [2, Theorem 1.3] as a corollary.

Combining Theorems 1.1 and 1.2, we immediately get the following Yaglom-type conditional limit theorem.

**Corollary 1.1.** *Assume* (1.2) *holds and $\xi$ is a Lévy process satisfying* **(H1)**. *Let $x > 0$.*

*(1) If $\mathbf{E}_0(\xi_1) = 0$, then we have*

$$\mathbb{P}_x\left(\frac{M_t}{\sqrt{t}} \in \cdot \Big| \zeta > t\right) \quad \overset{\mathrm{d}}{\Longrightarrow} \quad \mathcal{R}(\cdot),$$

*where $\mathcal{R}$ is the Rayleigh distribution with density $\rho(z) = z e^{-z^2/2} 1_{\{z > 0\}}$.*

*(2) If $\mathbf{E}_0(\xi_1) > 0$, then we have*

$$\mathbb{P}_x\left(\frac{M_t - \mathbf{E}_0(\xi_1) t}{\sqrt{t}} \in \cdot \Big| \zeta > t\right) \quad \overset{\mathrm{d}}{\Longrightarrow} \quad N(0, \sigma^2),$$

*where $N(0, \sigma^2)$ is the normal distribution with mean 0 and variance $\sigma^2$.*

*(3) If $\mathbf{E}_0(\xi_1) < 0$ and $\xi$ satisfies* **(H2)** *and* **(H3)**, *then there exists a random variable $(X, \mathbb{P})$ whose law is independent of $x$ such that*

$$\mathbb{P}_x\left(M_t \in \cdot \Big| \zeta > t\right) \quad \overset{\mathrm{d}}{\Longrightarrow} \quad \mathbb{P}(X \in \cdot).$$

In the following theorem we assume that $\xi$ is a spectrally negative Lévy process with Laplace exponent $\Psi$. For $q \geq 0$, let

$$\psi(q) := \sup\{\lambda \geq 0 : \Psi(\lambda) = q\}$$

be the right inverse of $\Psi$. By Kyprianou [12, Theorem 8.1], for any $q \geq 0$, there exists a scale function $W^{(q)} : \mathbb{R} \to [0, \infty)$ such that $W^{(q)}(x) = 0$ for $x < 0$ and $W^{(q)}$ is a strictly increasing and continuous function on $[0, \infty)$ with Laplace transform

$$\int_0^{\infty} e^{-rx} W^{(q)}(x) dx = \frac{1}{\Psi(r) - q}, \quad \text{for } r > \psi(q).$$

In the case when $\xi$ is a standard Brownian motion with drift $-b$, by using tables of Laplace transforms, one can easily get that

$$W^{(q)}(x) = \frac{2e^{bx}}{\sqrt{b^2 + 2q}} \sinh(\sqrt{b^2 + 2q} x), \quad x \geq 0, \ q \geq 0. \tag{1.13}$$

Our third main result is on the asymptotic behavior of the all-time maximum $M$ of branching killed spectrally negative Lévy process.

**Theorem 1.3.** *Assume that* (1.2) *holds and that $\xi$ is a spectrally negative Lévy process. There exists a constant $C_2(\alpha) \in (0, 1]$ such that for any $x > 0$,*

$$\lim_{y \to \infty} e^{\psi(\alpha) y} \mathbb{P}_x(M > y) = C_2(\alpha) W^{(\alpha)}(x) \Psi'(\psi(\alpha)),$$

*where $W^{(\alpha)}$ is the scale function of $((\xi_t)_{t \geq 0}, (\mathbf{P}_x)_{x \in \mathbb{R}})$.*

**Remark 4.** The reason we consider spectrally negative Lévy processes here, rather than general Lévy processes, is that the proof of Theorem 1.3 is closely related to the two-sided exit problem. For general Lévy processes, there are no tractable expressions for quantities of interest related to the two-sided exit problem. Combining the result above with (1.13), we immediately recover [2, Theorem 1.2] as a corollary. Profeta [6, Theorem 1.1] proved the following asymptotic behavior of the all-time maximum $\widetilde{M}$ for spectrally negative branching Lévy processes without killing

$$\mathbb{P}(\widetilde{M} \geq x) \sim \kappa e^{-\psi(\alpha) x}, \quad \text{as } x \to \infty, \tag{1.14}$$

under the third-moment condition on the offspring distribution $\{p_k\}_{k \geq 0}$, where $\kappa$ is a positive constant. Comparing Theorem 1.3 with (1.14), we observe that the killing barrier does not affect the exponential decay rate of the tail probability of the all-time maximum, it only affects the limits after the same exponential scaling.

### 1.3. Proof strategies and organization of the paper

The rest of the paper is organized as follows. In Section 2, we give some results on Lévy processes which will be used in the proofs of our main results. We establish the conditioned limit theorem for Lévy processes in Section 3. The proofs of Theorems 1.1 and 1.2 are given in Section 4, and the proof of Theorem 1.3 is given in Section 5.

Now we sketch the main idea of the proof of Theorem 1.1. The main idea for the proof of Theorem 1.2 is similar, and Corollary 1.1 follows from Theorems 1.1 and 1.2. For any $x, t > 0$, let

$$u(x, t) := \mathbb{P}_x(\zeta > t).$$

In Lemma 4.2, we derive a representation for $u(x, t)$. Lemma 4.3 then establishes a lower bound for $u(x, t)$, while Lemmas 4.4 and 4.5 provide upper bounds for $u(x, t)$ in the cases $\mathbf{E}_0(\xi_1) = 0$ and $\mathbf{E}_0(\xi_1) > 0$, respectively. Theorem 1.1 (1) and (2) follow immediately from the above lemmas. In the case $\mathbf{E}_0(\xi_1) < 0$, a quasi-stationary distribution exists, and the proof technique differs from those used in the previous two cases. The analysis of its asymptotic behavior relies on Theorem 3.1, which establishes a conditioned limit theorem for Lévy processes.

In this paper, we use $\phi(\cdot)$ to denote the standard normal density, i.e., $\phi(t) = \frac{1}{\sqrt{2\pi}} e^{-t^2/2}$, use $\rho(\cdot)$ to denote the Rayleigh density, i.e., $\rho(x) = x e^{-x^2/2} 1_{\{x>0\}}$, and use $\mathcal{R}(x)$ to denote the Rayleigh distribution function, i.e., $\mathcal{R}(x) = (1 - e^{-x^2/2}) 1_{\{x \geq 0\}}$. For $v > 0$, we define $\phi_v(x) = \frac{1}{\sqrt{2\pi v}} e^{-x^2/(2v)}$ and $\rho_v(x) = (x/v) e^{-x^2/(2v)} 1_{\{x>0\}}$. We use $F(x) \sim G(x)$ as $x \to \infty$ to denote $\lim_{x \to \infty} F(x)/G(x) = 1$. In this paper, capital letters $C_i$ and $T_i$, $i = 1, 2, \ldots$, are used to denote constants in the statements of results and their value remain the same throughout the paper. Lower case letters $c_i$, $i = 1, 2, \ldots$, are used for constants used in the proofs and their labeling starts anew in each proof. $c_i(\epsilon)$ and $C_i(\epsilon)$ mean that the constants $c_i$ and $C_i$ depend on $\epsilon$.

## 2. Preliminaries

In this section, we first present some preliminary results for spectrally negative Lévy processes, followed by a result for general Lévy processes. Assume for now that $\xi$ is a spectrally negative Lévy process with Laplace exponent $\Psi$. Then for any $x > 0$,

$$\mathbf{E}_0(\xi_{\tau_x^+} = x | \tau_x^+ < \infty) = 1.$$

Moreover, it is well known, see [12, Section 8], that for any $x > 0$ and $q \geq 0$,

$$\mathbf{E}_0\left(e^{-q\tau_x^+} 1_{\{\tau_x^+ < \infty\}}\right) = e^{-\psi(q)x},$$

where $\psi$ is the right inverse of $\Psi$. The following result on exit probabilities is contained in [12, Theorem 8.1].

**Theorem 2.1.** *Assume that $\xi$ is a spectrally negative Lévy process with Laplace exponent $\Psi$. For any $0 < x \leq y$ and $q \geq 0$,*

$$\mathbf{E}_x\left(e^{-q\tau_y^+} 1_{\{\tau_0^- > \tau_y^+\}}\right) = \frac{W^{(q)}(x)}{W^{(q)}(y)},$$

*where $W^{(q)}$ is the scale function of $\xi$.*

The following result, which can be found in [12, Lemma 8.4] and [13, Proposition 1], gives the relationship between $W_c^{(q)}$ for different values of $q$, $c$, and the asymptotic behavior of $W^{(q)}(x)$ as $x \to \infty$.

**Lemma 2.1.** *Assume that $\xi$ is a spectrally negative Lévy process with Laplace exponent $\Psi$. For any $x \geq 0$, the function $q \mapsto W^{(q)}(x)$ may be analytically extended to $q \in \mathbb{C}$. Furthermore, for any $q \in \mathbb{C}$ and $c \in \mathbb{R}$ with $\Psi(c) < \infty$, we have*

$$W^{(q)}(x) = e^{cx} W_c^{(q-\Psi(c))}(x), \quad x \geq 0,$$

*where $W_c^{(q-\Psi(c))}$ is the scale function of $\xi^{(c)}$. Furthermore,*

$$W^{(q)}(x) \sim \frac{e^{\psi(q)x}}{\Psi'(\psi(q))}, \quad as \ x \to \infty. \tag{2.1}$$

The following lemma is an important tool for proving Theorem 1.3.

**Lemma 2.2.** *Assume that $\xi$ is a spectrally negative Lévy process with Laplace exponent $\Psi$. For any $a > 0$, $0 < x \leq y$ and nonnegative Borel function $h$, we have*

$$\mathbf{E}_x\left(1_{\{\tau_y^+ < \tau_0^-\}} e^{-a\tau_y^+ - \int_0^{\tau_y^+} h(\xi_s)\mathrm{d}s}\right) = e^{\psi(a)(x-y)} \mathbf{E}_x^{\psi(a)}\left(1_{\{\tau_y^+ < \tau_0^-\}} e^{-\int_0^{\tau_y^+} h(\xi_s)\mathrm{d}s}\right).$$

**Proof.** By Theorem 6 on p16 of [14], $\{\tau_y^+ < \tau_0^-\} \cap \{\tau_y^+ < t\} = \{\tau_y^+ \wedge t < \tau_0^-\} \cap \{\tau_y^+ \wedge t < t\}$ is $\mathcal{F}_{\tau_y^+ \wedge t}$-measurable. For $a > 0$, since $e^{-a\tau_y^+} 1_{\{\tau_y^+ = \infty\}} = 0$, using (1.9) with $c = \psi(a)$, we have

$$
\begin{aligned}
\mathbf{E}_x\left(1_{\{\tau_y^+ < \tau_0^-\}} e^{-a\tau_y^+ - \int_0^{\tau_y^+} h(\xi_s)\mathrm{d}s}\right) &= \lim_{t\to\infty} \mathbf{E}_x\left(1_{\{\tau_y^+ < \tau_0^-, \tau_y^+ < t\}} e^{-a\tau_y^+ - \int_0^{\tau_y^+} h(\xi_s)\mathrm{d}s}\right) \\
&= \lim_{t\to\infty} \mathbf{E}_x^{\psi(a)}\left(e^{-\psi(a)(\xi_t - x) + at} e^{-a\tau_y^+ - \int_0^{\tau_y^+} h(\xi_s)\mathrm{d}s} 1_{\{\tau_y^+ < \tau_0^-, \tau_y^+ < t\}}\right) \\
&= \lim_{t\to\infty} \mathbf{E}_x^{\psi(a)}\left(e^{-a\tau_y^+ - \int_0^{\tau_y^+} h(\xi_s)\mathrm{d}s} 1_{\{\tau_y^+ < \tau_0^-, \tau_y^+ < t\}} \mathbf{E}_x^{\psi(a)}\left(e^{-\psi(a)(\xi_t - x) + at}\Big| \mathcal{F}_{\tau_y^+ \wedge t}\right)\right).
\end{aligned}
\tag{2.2}
$$

Note that $(e^{-\psi(a)(\xi_t - x) + at})_{t \geq 0}$ is a $\mathbf{P}_x^{\psi(a)}$ is a martingale with respect to $\mathcal{F}_t$. Using the optional stopping theorem and the absence of positive jumps, we get that, on $\{\tau_y^+ < t\}$,

$$
\mathbf{E}_x^{\psi(a)}\left(e^{-\psi(a)(\xi_t - x) + at}\Big| \mathcal{F}_{\tau_y^+ \wedge t}\right) = e^{-\psi(a)(\xi_{\tau_y^+ \wedge t} - x) + a(\tau_y^+ \wedge t)} = e^{-\psi(a)(y - x) + a\tau_y^+}.
$$

Combining this with (2.2) and using the fact that $\mathbf{P}_x^{\psi(a)}(\tau_y^+ < \infty) = 1$, we get

$$
\mathbf{E}_x\left(1_{\{\tau_y^+ < \tau_0^-\}} e^{-a\tau_y^+ - \int_0^{\tau_y^+} h(\xi_s)\mathrm{d}s}\right) = e^{\psi(a)(x - y)} \mathbf{E}_x^{\psi(a)}\left(1_{\{\tau_y^+ < \tau_0^-\}} e^{-\int_0^{\tau_y^+} h(\xi_s)\mathrm{d}s}\right).
$$

This gives the desired result. $\square$

The following lemma gives the joint asymptotic behavior of the tail of $\tau_0^-$ and the Lévy process $\xi$ when $\mathbf{E}_0(\xi_1) > 0$, and this result holds for general Lévy processes.

**Lemma 2.3.** *Assume that $\xi$ is a Lévy process such that $\mathbf{E}_0(\xi_1) > 0$ and $\sigma^2 := \mathbf{E}_0(\xi_1^2) < \infty$, then for any $x > 0$,*

$$
\lim_{t\to\infty} \mathbf{P}_x(\tau_0^- > t) = \mathbf{P}_x(\tau_0^- = \infty) =: q_x > 0.
\tag{2.3}
$$

*Moreover, for any $y \in \mathbb{R}$, we have*

$$
\lim_{t\to\infty} \mathbf{P}_x\left(\tau_0^- > t, \xi_t - \mathbf{E}_0(\xi_1)t > \sqrt{t}y\right) = \mathbf{P}_x(\tau_0^- = \infty) \int_{\frac{y}{\sigma}}^\infty \phi(z)\mathrm{d}z.
$$

**Proof.** Note that (2.3) follows immediately from [15, Proposition 17, p172]. Fix $t > 0$, for $m \in (0, t)$, by the Markov property,

$$
\mathbf{P}_x\left(\tau_0^- > t, \xi_t - \mathbf{E}_0(\xi_1)t > \sqrt{t}y\right) \leq \mathbf{P}_x\left(\tau_0^- > m, \xi_t - \mathbf{E}_0(\xi_1)t > \sqrt{t}y\right) = \mathbf{E}_x\left(1_{\{\tau_0^- > m\}} \mathbf{P}_{\xi_m}\left(\xi_{t-m} - \mathbf{E}_0(\xi_1)t > \sqrt{t}y\right)\right).
$$

By the central limit theorem, for any $z$, as $t \to \infty$, we get

$$
\mathbf{P}_z\left(\xi_{t-m} - \mathbf{E}_0(\xi_1)t > \sqrt{t}y\right) \to \int_y^\infty \phi_{\sigma^2}(u)\mathrm{d}u.
\tag{2.4}
$$

Letting $t \to \infty$ first, then $m \to \infty$, we get that

$$
\limsup_{t\to\infty} \mathbf{P}_x\left(\tau_0^- > t, \xi_t - \mathbf{E}_0(\xi_1)t > \sqrt{t}y\right) \leq \mathbf{P}_x(\tau_0^- = \infty) \int_y^\infty \phi_{\sigma^2}(z)\mathrm{d}z.
\tag{2.5}
$$

On the other hand, we have

$$
\mathbf{P}_x\left(\tau_0^- > m, \xi_t - \mathbf{E}_0(\xi_1)t > \sqrt{t}y\right) \leq \mathbf{P}_x\left(\tau_0^- > t, \xi_t - \mathbf{E}_0(\xi_1)t > \sqrt{t}y\right) + \mathbf{P}_x\left(\tau_0^- \in (m, t]\right).
$$

It follows from (2.4) that

$$
\begin{aligned}
\lim_{m\to\infty} \lim_{t\to\infty} \mathbf{P}_x\left(\tau_0^- > m, \xi_t - \mathbf{E}_0(\xi_1)t > \sqrt{t}y\right) &= \lim_{m\to\infty} \lim_{t\to\infty} \mathbf{E}_x\left(1_{\{\tau_0^- > m\}} \mathbf{P}_{\xi_m}\left(\xi_{t-m} - \mathbf{E}_0(\xi_1)t > \sqrt{t}y\right)\right) \\
&= \lim_{m\to\infty} \mathbf{P}_x(\tau_0^- > m) \int_y^\infty \phi_{\sigma^2}(u)\mathrm{d}u = \mathbf{P}_x(\tau_0^- = \infty) \int_y^\infty \phi_{\sigma^2}(u)\mathrm{d}u,
\end{aligned}
$$

this combined with

$$
\lim_{m\to\infty} \lim_{t\to\infty} \mathbf{P}_x\left(\tau_0^- \in (m, t]\right) = \lim_{m\to\infty} \mathbf{P}_x\left(\tau_0^- \in (m, \infty)\right) = 0
$$

yields that

$$
\liminf_{t\to\infty} \mathbf{P}_x\left(\tau_0^- > t, \xi_t - \mathbf{E}_0(\xi_1)t > \sqrt{t}y\right) \geq \mathbf{P}_x(\tau_0^- = \infty) \int_y^\infty \phi_{\sigma^2}(z)\mathrm{d}z.
$$

Combining this with (2.5), we get the desired result. $\square$

7

## 3. Conditioned limit theorems for Lévy processes

The purpose of this section is to prove Theorem 3.1, a conditioned limit theorem for Lévy processes. Theorem 3.1 will play an important role in this paper. We make some preparations first. The following result follows from [1, Lemmas 2.12 and 4.1].

**Lemma 3.1.** *Assume that $\xi$ is a Lévy process satisfying $\mathbf{E}_0(\xi_1) = 0$ and* **(H1)**. *Then for any $x > 0$ and $a \in (0, \infty]$, it holds that*

$$\lim_{t \to \infty} \sqrt{t} \, \mathbf{P}_x\left(\xi_t \le a\sqrt{t}, \tau_0^- > t\right) = \frac{2R(x)}{\sqrt{2\pi\sigma^2}} \int_0^{\frac{a}{\sigma}} \rho(z) \mathrm{d}z,$$

*where $\sigma^2 := \mathbf{E}_0(\xi_1^2)$ and $\rho(z)$ denotes the Rayleigh density. Furthermore, for any $x > 0$ and any bounded continuous function $h$ on $(0, \infty)$, it holds that*

$$\lim_{t \to \infty} \sqrt{t} \, \mathbf{E}_x\left( h\left(\frac{\xi_t}{\sigma\sqrt{t}}\right) 1_{\{\tau_0^- > t\}}\right) = \frac{2R(x)}{\sqrt{2\pi\sigma^2}} \int_0^\infty \rho(z) h(z) \mathrm{d}z.$$

Recall that $\delta$ is the constant in **(H1)** and $\sigma^2 = \mathbb{E}_0(\xi_1^2)$. The following result is [1, Lemma 2.11].

**Lemma 3.2.** *Assume that $\xi$ is a Lévy process satisfying $\mathbf{E}_0(\xi_1) = 0$ and* **(H1)**. *Then there exists a Brownian motion $W$ with diffusion coefficient $\sigma^2$ starting from the origin such that for any $\kappa \in (0, \frac{\delta}{2(2+\delta)})$, there exists a constant $C_3(\kappa) > 1$ such that for all $t \ge 1$,*

$$\mathbf{P}_x\left( \sup_{s \in [0,1]} |\xi_{ts} - x - W_{ts}| > t^{\frac{1}{2} - \kappa}\right) \le \frac{C_3(\kappa)}{t^{(\frac{1}{2} - \kappa)(\delta + 2) - 1}}.$$

The following lemma is a conditional limit theorem for Lévy processes. Its proof is similar to that of [1, Lemma 4.1], but it provides a more precise bound. See [11, Theorem 2.7] for an analogous result for random walks.

**Lemma 3.3.** *Assume that $\xi$ is a Lévy process satisfying $\mathbf{E}_0(\xi_1) = 0$, $\mathbf{E}_0(\xi_1^2) = \sigma^2$ and* **(H1)**. *Then one can find a constant $\varepsilon_0 \in (0, \frac{\delta}{4(2+\delta)})$ with the property that for any $\varepsilon \in (0, \varepsilon_0)$ there exist positive constants $T_0(\varepsilon)$ and $C_4(\varepsilon)$ such that for any $x, y > 0$ and $t > T_0(\varepsilon)$,*

$$\left| \mathbf{P}_x\left( \frac{\xi_t}{\sigma\sqrt{t}} \le y, \tau_0^- > t \right) - \frac{2R(x)}{\sigma\sqrt{2\pi t}} \mathcal{R}(y) \right| \le \frac{C_4(\varepsilon)(1 + x)}{t^{1/2 + \varepsilon}}.$$

**Proof.** Let $W$ be the Brownian motion in Lemma 3.2. For any $r > 0$ and $\epsilon \in (0, \delta/(4(5 + 2\delta)))$, define

$$A_r := \left\{ \sup_{s \in [0,1]} |\xi_{sr} - \xi_0 - W_{sr}| > r^{\frac{1}{2} - 2\epsilon} \right\}.$$

Let $(S_n)_{n \ge 0}$ be the random walk defined by $S_n := \xi_n$, $n \in \mathbb{N}$. For any $b \in \mathbb{R}$, define

$$\tau_b^{S,+} := \inf\{j \in \mathbb{N}, |S_j| > b\}.$$

By the Markov property, we have the following decomposition:

$$\mathbf{P}_x\left( \frac{\xi_t}{\sigma\sqrt{t}} \le y, \tau_0^- > t \right) = \sum_{k=1}^4 I_k,$$

where

$$I_1 := \mathbf{P}_x\left( \frac{\xi_t}{\sigma\sqrt{t}} \le y, \tau_0^- > t, \tau_{t^{1/2-\epsilon}}^{S,+} > [t^{1-\epsilon}] \right),$$

$$I_2 := \sum_{k=1}^{[t^{1-\epsilon}]} \mathbf{E}_x\left( \mathbf{P}_{\xi_k}\left( \frac{\xi_{t-k}}{\sigma\sqrt{t}} \le y, \tau_0^- > t - k, A_{t-k} \right); \tau_0^- > k, \tau_{t^{1/2-\epsilon}}^{S,+} = k \right),$$

$$I_3 := \sum_{k=1}^{[t^{1-\epsilon}]} \mathbf{E}_x\left( \mathbf{P}_{\xi_k}\left( \frac{\xi_{t-k}}{\sigma\sqrt{t}} \le y, \tau_0^- > t - k, A_{t-k}^c \right); \tau_0^- > k, \xi_k > t^{(1-\epsilon)/2}, \tau_{t^{1/2-\epsilon}}^{S,+} = k \right),$$

$$I_4 := \sum_{k=1}^{[t^{1-\epsilon}]} \mathbf{E}_x\left( \mathbf{P}_{\xi_k}\left( \frac{\xi_{t-k}}{\sigma\sqrt{t}} \le y, \tau_0^- > t - k, A_{t-k}^c \right); \tau_0^- > k, \xi_k \le t^{(1-\epsilon)/2}, \tau_{t^{1/2-\epsilon}}^{S,+} = k \right).$$

We now deal with $I_i$, $i = 1, 2, 3, 4$, separately.

(i) Upper bound of $I_1$. Set $K := [t^\epsilon - 1]$ and $l := [t^{1-2\epsilon}]$. Since $Kl \le [t^{1-\epsilon}]$, we have

$$I_1 \le \mathbf{P}_x\left( \tau_{t^{1/2-\epsilon}}^{S,+} > [t^{1-\epsilon}] \right) \le \mathbf{P}_0\left( \max_{1 \le j \le Kl} |x + S_j| \le t^{1/2-\epsilon} \right) \le \mathbf{P}_0\left( \max_{1 \le j \le K} |x + S_{lj}| \le t^{1/2-\epsilon} \right), \quad x > 0. \tag{3.1}$$

By the Markov property, we have

$$\mathbf{P}_0\left(\max_{1 \le j \le K} |x + S_{lj}| \le t^{1/2-\epsilon}\right) \le \left(\sup_{x \in \mathbb{R}_+} \mathbf{P}_0\left(|x + S_l| \le t^{1/2-\epsilon}\right)\right)^K. \tag{3.2}$$

According to the display below [1, (4.6)], there exist positive constants $c_1 \in (0,1)$ and $t_1(\epsilon)$ such that for $t > t_1(\epsilon)$,

$$\mathbf{P}_0\left(|x + S_l| \le t^{1/2-\epsilon}\right) < c_1 \quad x \in \mathbb{R}_+.$$

Plugging this into (3.2), taking $c_2 = -\ln c_1$, and combining with (3.1), we get that for $t > t_1(\epsilon)$,

$$I_1 \le c_1^K = e^{-c_2[t^\epsilon - 1]} \le \frac{c_2}{t^{1/2 + \epsilon/8}}. \tag{3.3}$$

(ii) Upper bound of $I_2$. By part (ii) of the proof of [1, Lemma 4.1], for any $\epsilon \in (0, \delta/(4(5+2\delta)))$, we have

$$I_2 \le \frac{C_3(2\epsilon)x}{t^{1/2 + \delta/2 - (5+2\delta)\epsilon}} \le \frac{C_3(2\epsilon)x}{t^{\frac{1}{2} + \epsilon/8}}, \tag{3.4}$$

where $C_3(2\epsilon)$ is the constant in Lemma 3.2.

(iii) Upper bound of $I_3$. Repeating the argument in part (iii) of the proof of [1, Lemma 4.1] leading to [1, (4.8)] and using [16, Lemma 7.7], we can find $\epsilon_1 > 0$ with the property that for any $\epsilon \in (0, \epsilon_1 \wedge \delta/(4(5+2\delta)))$ there exists a positive constant $c_3(\epsilon)$ such that

$$I_3 \le \frac{1}{\sqrt{t}} \sum_{k=1}^{[t^{1-\epsilon}]} \mathbf{E}_x\left(S_k; \tau_0^{S,-} > k, S_k > t^{(1-\epsilon)/2}, \tau_{t^{1/2-\epsilon}}^{S,+} = k\right) \le \frac{c_3(\epsilon)(1+x)}{t^{1+\delta/2 - \epsilon(1+\epsilon+\delta/2)}} \le \frac{c_3(\epsilon)(1+x)}{t^{\frac{1}{2} + \epsilon/8}}. \tag{3.5}$$

(iv) Upper bound of $I_4$. For $k \le [t^{1-\epsilon}]$ and $x' > 0$, define

$$K(k, x') := \mathbf{P}_{x'}\left(\frac{\xi_{t-k}}{\sigma\sqrt{t}} \le y, \tau_0^- > t - k, A_{t-k}^c\right).$$

Set

$$x^* := \frac{x' + (t-k)^{\frac{1}{2} - 2\epsilon}}{\sigma} \quad \text{and} \quad y^* := \frac{y\sqrt{t}}{\sqrt{t-k}} + \frac{2}{\sigma(t-k)^{2\epsilon}}.$$

It follows from [1, (4.13)] that

$$K(k, x') \le \frac{2}{\sqrt{2\pi(t-k)}}\left(\frac{x'}{\sigma} + \frac{t^{\frac{1}{2} - 2\epsilon}}{\sigma}\right) \int_0^{y^*} \rho(z) e^{\frac{zx^*}{\sqrt{t-k}}} \, \mathrm{d}z. \tag{3.6}$$

We claim that there exist positive constants $t_2(\epsilon)$, $c_4(\epsilon)$ such that for $t > t_2(\epsilon)$ and $p \ge 2$ sufficiently large,

$$K(k, x') \le \frac{2}{\sigma\sqrt{2\pi t}}\left(1 + \frac{c_4(\epsilon)}{t^{\epsilon/2 - \epsilon^p}}\right)\left(\mathcal{R}(y) + \frac{c_4(\epsilon)}{t^{\epsilon/2}}\right)\left(x' + t^{\frac{1}{2} - 2\epsilon}\right). \tag{3.7}$$

To prove this claim, note that for any $k \le [t^{1-\epsilon}]$ and $x' \le t^{(1-\epsilon)/2}$, there exist positive constants $t_3(\epsilon)$, $c_5(\epsilon)$ and $c_6(\epsilon)$ such that for $t > t_3(\epsilon)$, the following holds:

$$\frac{x^*}{\sqrt{t-k}} \le c_5(\epsilon)\frac{t^{(1-\epsilon)/2} + t^{\frac{1}{2} - 2\epsilon}}{\sqrt{t}} \le c_6(\epsilon)t^{-\epsilon/2}.$$

For $y \in [0, t^{\epsilon^p}]$ with $p \ge 2$ being a positive constant, and $z \le y^*$, there exist positive constants $t_4(\epsilon)$ and $c_7(\epsilon)$ such that for $t > t_4(\epsilon)$,

$$z \le \frac{y\sqrt{t}}{\sqrt{t-k}} + \frac{2}{\sigma(t-k)^{2\epsilon}} \le c_7(\epsilon)t^{\epsilon^p},$$

and thus there exists a positive constant $c_8(\epsilon)$ such that for $t > t_3(\epsilon) \vee t_4(\epsilon)$,

$$e^{\frac{zx^*}{\sqrt{t-k}}} \le e^{c_6(\epsilon)c_7(\epsilon)t^{-\epsilon/2}t^{\epsilon^p}} \le 1 + \frac{c_8(\epsilon)}{t^{\epsilon/2 - \epsilon^p}}, \quad \text{for } z \le y^*.$$

This implies that when $y \in [0, t^{\epsilon^p}]$, for $t > t_3(\epsilon) \vee t_4(\epsilon)$, it holds that

$$\int_0^{y^*} \rho(z) e^{\frac{zx^*}{\sqrt{t-k}}} \, \mathrm{d}z \le \left(1 + \frac{c_8(\epsilon)}{t^{\epsilon/2 - \epsilon^p}}\right) \int_0^{y^*} \rho(z) \mathrm{d}z.$$

Moreover, by the definition of $y^*$, there exist positive constants $t_5(\epsilon)$ and $c_9(\epsilon)$ such that for $t > t_5(\epsilon)$,

$$y^* - y \le \frac{c_9(\epsilon)}{t^{\epsilon/2}}.$$

Thus using the fact that $\rho(z) \leq 1$ for all $z \geq 0$, we get that for any $\epsilon \in (0, \epsilon_1 \wedge \delta/(4(5+2\delta)))$ and $t > \max\{t_i(\epsilon) : 3 \leq i \leq 5\}$,

$$\int_0^{y^*} \rho(z) e^{\frac{zx^*}{\sqrt{t-k}}} \, dz \leq \left(1 + \frac{c_8(\epsilon)}{t^{\epsilon/2 - \epsilon^p}}\right)\left(\mathcal{R}(y) + y^* - y\right) \leq \left(1 + \frac{c_8(\epsilon)}{t^{\epsilon/2 - \epsilon^p}}\right)\left(\mathcal{R}(y) + \frac{c_9(\epsilon)}{t^{\epsilon/2}}\right). \tag{3.8}$$

For $y > t^{\epsilon^p}$, using [16, (7.31)] we get that there exist positive constants $t_6(\epsilon)$ and $c_{10}(\epsilon)$ such that for $t > t_6(\epsilon)$,

$$\int_0^{y^*} \rho(z) e^{\frac{zx^*}{\sqrt{t-k}}} \, dz \leq \left(1 + \frac{c_{10}(\epsilon)}{t^{\epsilon/2 - \epsilon^p}}\right) \int_0^y \rho(z) dz + c_{10}(\epsilon) e^{-c_{10}(\epsilon) t^{\epsilon^p}} \leq \left(1 + \frac{c_{10}(\epsilon)}{t^{\epsilon/2 - \epsilon^p}}\right)\left(\mathcal{R}(y) + \frac{c_{10}(\epsilon)}{t^{\epsilon}}\right). \tag{3.9}$$

Combining (3.6), (3.8) and (3.9), we get that there exists a positive constant $c_{11}(\epsilon)$ such that for any $y > 0$ and $t > \max\{t_i(\epsilon) : 3 \leq i \leq 6\}$, we have

$$K(k, x') \leq \frac{2x^*}{\sqrt{2\pi(t-k)}}\left(1 + \frac{c_{11}(\epsilon)}{t^{\epsilon/2 - \epsilon^p}}\right)\left(\mathcal{R}(y) + \frac{c_{11}(\epsilon)}{t^{\epsilon/2}}\right). \tag{3.10}$$

Since $k \leq [t^{1-\epsilon}]$, there exists a constant $t_7(\epsilon) > 0$ such that when $t > t_7(\epsilon)$,

$$\frac{1}{\sqrt{t-k}} \leq \frac{1}{\sqrt{t}}\left(1 + \frac{c_{12}(\epsilon)}{t^{\epsilon}}\right),$$

and

$$\frac{x^*}{\sqrt{t-k}} \leq \frac{1}{\sigma\sqrt{t}}\left(1 + \frac{c_{13}(\epsilon)}{t^{\epsilon}}\right)\left(x' + t^{\frac{1}{2} - 2\epsilon}\right), \tag{3.11}$$

for some positive constants $c_{12}(\epsilon)$ and $c_{13}(\epsilon)$. Taking $t_2(\epsilon) := \max\{t_i(\epsilon) : 3 \leq i \leq 7\}$, then the claim (3.7) follows from (3.10) and (3.11).

Note that on $\{\tau^{S,+}_{t^{1/2-\epsilon}} = k\}$, we have $\xi_k = S_k \geq t^{1/2-\epsilon}$. Thus, $t^{1/2-\epsilon} \leq t^{-\epsilon}\xi_k$ on $\{\tau^{S,+}_{t^{1/2-\epsilon}} = k\}$. Also note that, by using that $\left(R(\xi_s)1_{\{\tau_0^- > s\}}\right)_{s \geq 0}$ is a $\mathbf{P}_x$-martingale for any $x > 0$ and the optional stopping theorem,

$$R(x) = \mathbf{E}_x\left(R(\xi_{\tau^{S,+}_{t^{1/2-\epsilon}}}); \tau_0^- > \tau^{S,+}_{t^{1/2-\epsilon}}\right), \quad x > 0, \ t > 0. \tag{3.12}$$

Hence, by (3.7), for $t > t_2(\epsilon)$,

$$I_4 \leq \frac{2}{\sigma\sqrt{2\pi t}}\left(1 + \frac{c_4(\epsilon)}{t^{\epsilon/2 - \epsilon^p}}\right)\left(\mathcal{R}(y) + \frac{c_4(\epsilon)}{t^{\epsilon/2}}\right)\sum_{k=1}^{[t^{1-\epsilon}]} \mathbf{E}_x\left(\xi_k + t^{\frac{1}{2} - 2\epsilon}; \tau_0^- > k, \xi_k \leq t^{(1-\epsilon)/2}, \tau^{S,+}_{t^{1/2-\epsilon}} = k\right)$$

$$\leq \frac{2(1 + t^{-\epsilon})}{\sigma\sqrt{2\pi t}}\left(1 + \frac{c_4(\epsilon)}{t^{\epsilon/2 - \epsilon^p}}\right)\left(\mathcal{R}(y) + \frac{c_4(\epsilon)}{t^{\epsilon/2}}\right)\sum_{k=1}^{[t^{1-\epsilon}]} \mathbf{E}_x\left(\xi_k; \tau_0^- > k, \xi_k \leq t^{(1-\epsilon)/2}, \tau^{S,+}_{t^{1/2-\epsilon}} = k\right)$$

$$\leq \frac{2(1 + t^{-\epsilon})}{\sigma\sqrt{2\pi t}}\left(1 + \frac{c_4(\epsilon)}{t^{\epsilon/2 - \epsilon^p}}\right)\left(\mathcal{R}(y) + \frac{c_4(\epsilon)}{t^{\epsilon/2}}\right)\mathbf{E}_x\left(\xi_{\tau^{S,+}_{t^{1/2-\epsilon}}}; \tau_0^- > \tau^{S,+}_{t^{1/2-\epsilon}}, \tau^{S,+}_{t^{1/2-\epsilon}} \leq [t^{1-\epsilon}]\right)$$

$$\leq \frac{2(1 + t^{-\epsilon})}{\sigma\sqrt{2\pi t}}\left(1 + \frac{c_4(\epsilon)}{t^{\epsilon/2 - \epsilon^p}}\right)\left(\mathcal{R}(y) + \frac{c_4(\epsilon)}{t^{\epsilon/2}}\right)\mathbf{E}_x\left(\xi_{\tau^{S,+}_{t^{1/2-\epsilon}}}; \tau_0^- > \tau^{S,+}_{t^{1/2-\epsilon}}\right)$$

$$= \frac{2R(x)(1 + t^{-\epsilon})}{\sigma\sqrt{2\pi t}}\left(1 + \frac{c_4(\epsilon)}{t^{\epsilon/2 - \epsilon^p}}\right)\left(\mathcal{R}(y) + \frac{c_4(\epsilon)}{t^{\epsilon/2}}\right),$$

where in the last equality we used (3.12). Thus, there exist positive constants $t_8(\epsilon)$, $c_{14}(\epsilon)$ and $c_{15}(\epsilon)$ such that for $t > t_8(\epsilon)$,

$$I_4 \leq \frac{2R(x)\left(1 + \frac{c_{14}(\epsilon)}{t^{\epsilon/2 - \epsilon^p}}\right)\left(\mathcal{R}(y) + \frac{c_{14}(\epsilon)}{t^{\epsilon/2}}\right)}{\sigma\sqrt{2\pi t}} \leq \frac{2R(x)\mathcal{R}(y)}{\sigma\sqrt{2\pi t}} + \frac{c_{15}(\epsilon)(1+x)}{t^{\frac{1}{2} + \epsilon/8}}, \tag{3.13}$$

where in the last inequality we use the fact that $R(x) \leq c(1+x)$ for some constant $c > 0$.

(v) Lower bound of $I_4$. Repeating the proof of [16, (7.40)], we get that there exist positive constants $t_9(\epsilon)$, $c_{16}(\epsilon)$ and $c_{17}(\epsilon)$ such that for $t > t_9(\epsilon)$,

$$I_4 \geq \frac{2R(x)}{\sigma\sqrt{2\pi t}}\left(1 - \frac{c_{16}(\epsilon)}{t^{\epsilon/2 - \epsilon^p}}\right)\left(\mathcal{R}(y) - \frac{1}{t^{2\epsilon}}\right) - \frac{c_{16}(\epsilon)(1+x)}{t^{\delta/2 - \epsilon(1+\epsilon+\delta/2)}} \geq \frac{2R(x)\mathcal{R}(y)}{\sigma\sqrt{2\pi t}} - \frac{c_{17}(\epsilon)(1+x)}{t^{\frac{1}{2} + \epsilon/8}}. \tag{3.14}$$

Set $\epsilon_0 := \min\{\delta/(4(5+2\delta)), \epsilon_1\}$, $\varepsilon := \epsilon/8$, $\varepsilon_0 := \epsilon_0/8$ and $T_0(\epsilon) := \max\{t_i(\epsilon) : i = 1, 2, 8, 9\}$. Using the fact that there exists $c_{18} > 0$ such that $R(x) \leq c_{18}(1+x)$, and combining (3.3), (3.4), (3.5), (3.13) and (3.14), we arrive at the conclusion of the lemma. □

The duality relations in the following lemma, especially (3.15), are well known in probabilistic potential theory. We give an elementary proof here for the reader's convenience.

**Lemma 3.4.** *For any $t > 0$ and any bounded Borel functions $g, h : \mathbb{R} \to \mathbb{R}_+$, we have*

$$\int_{\mathbb{R}_+} h(x)\mathbf{E}_x\left(g(\xi_t)1_{\{\tau_0^- > t\}}\right)dx = \int_{\mathbb{R}_+} g(y)\mathbf{E}_y\left(h(\widehat{\xi}_t)1_{\{\widehat{\tau}_0^- > t\}}\right)dy \tag{3.15}$$

*and*

$$\int_{\mathbb{R}} h(x)\mathbf{E}_x\Big(g(\xi_t)1_{\{\tau_0^- \leq t\}}\Big)dx = \int_{\mathbb{R}} g(y)\mathbf{E}_y\Big(h(\widehat{\xi}_t)1_{\{\widehat{\tau}_0^- \leq t\}}\Big)dy. \tag{3.16}$$

**Proof.** For $x > 0$, by the change of variables $x + \xi_t = y$, we get

$$\int_{\mathbb{R}_+} h(x)\mathbf{E}_x\Big(g(\xi_t)1_{\{\tau_0^- > t\}}\Big)dx = \int_{\mathbb{R}_+} h(x)\mathbf{E}\Big(g(x + \xi_t)1_{\{\tau_{-x}^- > t\}}\Big)dx = \int_{\mathbb{R}_+} h(x)\mathbf{E}\Big(g(x + \xi_t), \inf_{s \leq t}\xi_s > -x\Big)dx$$

$$= \int_{\mathbb{R}_+} h(x)\mathbf{E}\Big(g(x + \xi_t), \inf_{s \leq t}\xi_{t-s} > -x\Big)dx = \int_{\mathbb{R}_+} g(y)\mathbf{E}\Big(h(y - \xi_t), \inf_{s \leq t}(\xi_{t-s} - \xi_t) > -y\Big)dy$$

$$= \int_{\mathbb{R}_+} g(y)\mathbf{E}\Big(h(y + \widehat{\xi}_t), \inf_{s \leq t}\widehat{\xi}_s > -y\Big)dy = \int_{\mathbb{R}_+} g(y)\mathbf{E}_y\Big(h(\widehat{\xi}_t)1_{\{\widehat{\tau}_0^- > t\}}\Big)dy,$$

which completes the proof of (3.15). Using the same argument, we can also get

$$\int_{\mathbb{R}} h(x)\mathbf{E}_x(g(\xi_t))dx = \int_{\mathbb{R}} h(x)\mathbf{E}(g(x + \xi_t))dx = \int_{\mathbb{R}} g(y)\mathbf{E}(h(y - \xi_t))dy = \int_{\mathbb{R}} g(y)\mathbf{E}(h(y + \widehat{\xi}_t))dy = \int_{\mathbb{R}} g(y)\mathbf{E}_y(h(\widehat{\xi}_t))dy. \tag{3.17}$$

Note that for $x < 0$, $\mathbf{P}_x(\tau_0^- > t) = \mathbf{P}_x(\widehat{\tau}_0^- > t) = 0$. Therefore, (3.15) is equivalent to

$$\int_{\mathbb{R}} h(x)\mathbf{E}_x\Big(g(\xi_t)1_{\{\tau_0^- > t\}}\Big)dx = \int_{\mathbb{R}} g(y)\mathbf{E}_y\Big(h(\widehat{\xi}_t)1_{\{\widehat{\tau}_0^- > t\}}\Big)dy.$$

Combining this with (3.17), we get (3.16). $\square$

Before stating Theorem 3.1, we first introduce some necessary notation and definitions. Let $h_1, h_2 : \mathbb{R} \to \mathbb{R}_+$ be Borel functions and $\varepsilon > 0$. We say that $h_1$ $\varepsilon$-dominates $h_2$ and write $h_2 \leq_\varepsilon h_1$ if

$$h_2(u) \leq h_1(u + v), \quad \forall u \in \mathbb{R}, \ \forall v \in [-\varepsilon, \varepsilon].$$

For any $a > 0$ and Borel function $h : \mathbb{R} \to \mathbb{R}_+$, we define $I_{k,a} = [ka, (k+1)a]$ for $k \in \mathbb{Z}$ and

$$\bar{h}_a(u) := \sum_{k \in \mathbb{Z}} 1_{I_{k,a}}(u) \sup_{u' \in I_{k,a}} f(u'), \quad \underline{h}_a(u) := \sum_{k \in \mathbb{Z}} 1_{I_{k,a}}(u) \inf_{u' \in I_{k,a}} f(u'), \quad u \in \mathbb{R}.$$

The function $h$ is called directly Riemann integrable if $\int_{\mathbb{R}} \bar{h}_a(u)du < \infty$ for any $a > 0$ small enough and

$$\lim_{a \to 0} \int_{\mathbb{R}} (\bar{h}_a(u) - \underline{h}_a(u))du = 0.$$

Define

$$\bar{h}_{a,\varepsilon}(u) := \sup_{[u-\varepsilon, u+\varepsilon]} \bar{h}_a(v), \quad \underline{h}_{a,-\varepsilon}(u) := \inf_{v \in [u-\varepsilon, u+\varepsilon]} \bar{h}_a(v), \quad u \in \mathbb{R}, \tag{3.18}$$

then it holds that

$$\underline{h}_{a,-\varepsilon} \leq_\varepsilon \underline{h}_a \leq h \leq \bar{h}_a \leq_\varepsilon \bar{h}_{a,\varepsilon} \quad \text{on } \mathbb{R}.$$

For more details about directly Riemann integrability, see [17, Section XI.1].

The following theorem will play an important role in this paper. We refer the reader to [11, Theorem 1.9] for an analogous result for random walks.

**Theorem 3.1.** *Assume that $\xi$ is a Lévy process satisfying* **(H1)**, **(H2)**, **(H3)** *and* $\mathbf{E}_0(\xi_1) < 0$. *Let* $f : \mathbb{R} \to \mathbb{R}_+$ *be a Borel function, which is not 0 almost everywhere on* $\mathbb{R}_+$, *such that* $f(x)e^{-\lambda_* x}(1 + |x|)$ *is directly Riemann integrable. Then for any $x > 0$, it holds that*

$$\lim_{t \to \infty} t^{3/2} e^{-\Psi(\lambda_*)t}\mathbf{E}_x\big(f(\xi_t), \tau_0^- > t\big) = \frac{2R^*(x)e^{\lambda_* x}}{\sqrt{2\pi\Psi''(\lambda_*)^3}} \int_{\mathbb{R}_+} f(z)e^{-\lambda_* z}\widehat{R}^*(z)dz,$$

*where $\Psi$ is the Laplace exponent of $\xi$.*

**Remark 5.** Recall that when $\mathbf{E}_0(\xi_1) < 0$ and **(H3)** holds, $\xi^{(\lambda_*)}$ is a Lévy process with Laplace exponent $\Psi_{\lambda_*}(\lambda) = \Psi(\lambda + \lambda_*) - \Psi(\lambda_*)$ and that $\Psi'_{\lambda_*}(0+) = \Psi'(\lambda_*) = 0$. Using (1.10), we get that

$$\mathbf{E}_x\big(f(\xi_t), \tau_0^- > t\big) = e^{\Psi(\lambda_*)t + \lambda_* x}\mathbf{E}_x^{\lambda_*}\big(f(\xi_t)e^{-\lambda_* \xi_t}, \tau_0^- > t\big). \tag{3.19}$$

Therefore, to get the assertion of Theorem 3.1, we only need to consider the asymptotic behavior of

$$\mathbf{E}_x^{\lambda_*}\big(f(\xi_t)e^{-\lambda_* \xi_t}, \tau_0^- > t\big), \quad \text{as } t \to \infty.$$

**Theorem 3.2.** *Assume that $\xi$ is a Lévy process satisfying* **(H1)**, **(H2)**, **(H3)** *and* $\mathbf{E}_0(\xi_1) < 0$. *Let* $h : \mathbb{R} \to \mathbb{R}_+$ *be a Borel function, which is not 0 almost everywhere on* $\mathbb{R}_+$, *such that* $h(x)(1 + |x|)$ *is directly Riemann integrable. Then for any $x > 0$, it holds that*

$$\lim_{t \to \infty} t^{3/2}\mathbf{E}_x^{\lambda_*}\big(h(\xi_t)1_{\{\tau_0^- > t\}}\big) = \frac{2R^*(x)}{\sqrt{2\pi\Psi''(\lambda_*)^3}} \int_{\mathbb{R}_+} h(z)\widehat{R}^*(z)dz,$$

*where $\Psi$ is the Laplace exponent of $\xi$.*

**Remark 6.** The difference between the asymptotic behavior in Theorem 3.2 (with $t^{-3/2}$ decay) and that in Lemma 3.1 (with $t^{-1/2}$ decay) arises from the fact that Theorem 3.2 is a conditioned limit theorem for the process $(\xi_t)_{t\geq 0}$ itself, whereas Lemma 3.1 is a conditioned limit result for the normalized process $\left(\frac{\xi_t}{\sigma\sqrt{t}}\right)_{t\geq 0}$.

**Proof of Theorem 3.1** Taking $h(x) = f(x)e^{-\lambda_* x}$ in Theorem 3.2 and using (3.19), we immediately get the conclusion of Theorem 3.1. $\square$

In the next four lemmas, we provide some upper and lower bounds for $\mathbf{E}_x^{\lambda_*}\left(h(\xi_t)1_{\{\tau_0^->t\}}\right)$. In the remainder of this section, $\epsilon_0$ will be the constant in Lemma 3.3. Recall that $\rho(\cdot)$ stands for the Rayleigh density.

**Lemma 3.5.** *Assume that $\xi$ is a Lévy process satisfying* **(H1)**, **(H2)**, **(H3)** *and* $\mathbf{E}_0(\xi_1) < 0$. *Then one can find a constant $C_5 > 0$ with the property that for any $\epsilon \in (0, \epsilon_0)$ there exist positive constants $T_1(\epsilon)$ and $C_6(\epsilon)$ such that for any $x > 0$, $t > T_1(\epsilon)$ and any integrable functions $h, H : \mathbb{R} \to \mathbb{R}_+$ satisfying $h \leq_\epsilon H$,*

$$\mathbf{E}_x^{\lambda_*}\left(h(\xi_t)1_{\{\tau_0^->t\}}\right) \leq \frac{2(1+C_5\epsilon)R^*(x)}{\sqrt{2\pi}\Psi''(\lambda_*)t} \int_{\mathbb{R}_+} H(w)\rho\left(\frac{w}{\sqrt{\Psi''(\lambda_*)t}}\right)\mathrm{d}w + \frac{2C_5\sqrt{\epsilon}R^*(x)}{\Psi''(\lambda_*)t} \int_{-\epsilon}^\infty H(w)\phi\left(\frac{w}{\sqrt{\Psi''(\lambda_*)t}}\right)\mathrm{d}w$$
$$+ C_6(\epsilon)(1+x)\|H1_{[-\epsilon,\infty)}\|_1\left(\frac{1}{t^{1+\epsilon}} + \frac{1}{t^{1+\delta/2}}\right),$$

*where $\Psi$ is the Laplace exponent of $\xi$.*

**Proof.** Fix $\epsilon \in (0, \epsilon_0)$ and let $h, H : \mathbb{R} \to \mathbb{R}_+$ be integrable functions satisfying $h \leq_\epsilon H$. Fix $t \geq 1$ and set $m = [\epsilon t]$. By the Markov property,

$$\mathbf{E}_x^{\lambda_*}\left(h(\xi_t)1_{\{\tau_0^->t\}}\right) = \int_{\mathbb{R}_+} \mathbf{E}_y^{\lambda_*}\left(h(\xi_m)1_{\{\tau_0^->m\}}\right)\mathbf{P}_x^{\lambda_*}(\xi_{t-m} \in \mathrm{d}y, \tau_0^- > t - m). \tag{3.20}$$

Define a random walk $(S_n)_{n\geq 0}$ by $S_n := \xi_n$, $n \in \mathbb{N}$. Since $h1_{[0,\infty)} \leq_\epsilon H1_{[-\epsilon,\infty)}$, it follows from [11, Theorem 2.7] that there exist constants $c_1$ (independent of $\epsilon$) and $c_2(\epsilon)$ such that for any $n \geq 1$,

$$\mathbf{E}_x^{\lambda_*}\left(h(S_n)1_{\{S_n\geq 0\}}\right) - \frac{1+c_1\epsilon}{\sqrt{\Psi''(\lambda_*)n}} \int_{\mathbb{R}} H(z)1_{\{z\geq-\epsilon\}}\phi\left(\frac{z-x}{\sqrt{\Psi''(\lambda_*)n}}\right)\mathrm{d}z \leq \frac{c_2(\epsilon)}{n^{(1+\delta)/2}}\|H1_{[-\epsilon,\infty)}\|_1. \tag{3.21}$$

Thus, for any $y > 0$,

$$\mathbf{E}_y^{\lambda_*}\left(h(\xi_m)1_{\{\tau_0^->m\}}\right) \leq \mathbf{E}_y^{\lambda_*}\left(h(S_m)1_{\{S_m\geq 0\}}\right) \leq \frac{1+c_1\epsilon}{\sqrt{\Psi''(\lambda_*)m}} \int_{\mathbb{R}} H(z)1_{\{z\geq-\epsilon\}}\phi\left(\frac{z-y}{\sqrt{\Psi''(\lambda_*)m}}\right)\mathrm{d}z + \frac{c_2(\epsilon)}{m^{(1+\delta)/2}}\|H1_{[-\epsilon,\infty)}\|_1.$$

Plugging this into (3.20) yields that

$$\mathbf{E}_x^{\lambda_*}\left(h(\xi_t)1_{\{\tau_0^->t\}}\right) \leq \int_{\mathbb{R}_+} \left(\frac{1+c_1\epsilon}{\sqrt{\Psi''(\lambda_*)m}} \int_{\mathbb{R}} H(z)1_{\{z\geq-\epsilon\}}\phi\left(\frac{z-y}{\sqrt{\Psi''(\lambda_*)m}}\right)\mathrm{d}z\right)\mathbf{P}_x^{\lambda_*}(\xi_{t-m} \in \mathrm{d}y, \tau_0^- > t - m) \tag{3.22}$$
$$+ \int_{\mathbb{R}_+} \frac{c_2(\epsilon)\|H1_{[-\epsilon,\infty)}\|_1}{m^{(1+\delta)/2}}\mathbf{P}_x^{\lambda_*}(\xi_{t-m} \in \mathrm{d}y, \tau_0^- > t - m)$$
$$=: A_1(x) + A_2(x).$$

By the definition of $\tau_0^-$, we have

$$\mathbf{P}_x^{\lambda_*}(\tau_0^- > s) = \mathbf{P}_x^{\lambda_*}\left(\inf_{l\leq s}\xi_l > 0\right) \leq \mathbf{P}_x^{\lambda_*}\left(\inf_{j\leq[s]}S_j > 0\right) \leq c_3\frac{1+x}{\sqrt{s}}, \tag{3.23}$$

for some positive constant $c_3$ (independent of $\epsilon$), where in the last inequality we used [18, (2.7)]. Therefore, by (3.23) and the definition of $m$, there exists a positive constant $c_4(\epsilon)$ such that

$$A_2(x) = \frac{c_2(\epsilon)\|H1_{[-\epsilon,\infty)}\|_1}{m^{(1+\delta)/2}}\mathbf{P}_x^{\lambda_*}(\tau_0^- > t - m) \leq \frac{c_4(\epsilon)(1+x)}{t^{1+\delta/2}}\|H1_{[-\epsilon,\infty)}\|_1. \tag{3.24}$$

Now, by a change of variables, we get

$$A_1(x) = \int_{\mathbb{R}_+} \left(\frac{1+c_1\epsilon}{\sqrt{\Psi''(\lambda_*)m}} \int_{\mathbb{R}} H(z)1_{\{z\geq-\epsilon\}}\phi\left(\frac{z-\sqrt{\Psi''(\lambda_*)(t-m)}u}{\sqrt{\Psi''(\lambda_*)m}}\right)\mathrm{d}z\right)\mathbf{P}_x^{\lambda_*}\left(\frac{\xi_{t-m}}{\sqrt{\Psi''(\lambda_*)(t-m)}} \in \mathrm{d}u, \tau_0^- > t - m\right)$$
$$= \int_{\mathbb{R}_+} \varphi_t(u)\mathbf{P}_x^{\lambda_*}\left(\frac{\xi_{t-m}}{\sqrt{\Psi''(\lambda_*)(t-m)}} \in \mathrm{d}u, \tau_0^- > t - m\right),$$

where the function $\varphi_t$ is defined by

$$\varphi_t(u) := \frac{1+c_1\epsilon}{\sqrt{\Psi''(\lambda_*)m}} \int_{\mathbb{R}} H(z)1_{\{z\geq-\epsilon\}}\phi\left(\frac{z-\sqrt{\Psi''(\lambda_*)(t-m)}u}{\sqrt{\Psi''(\lambda_*)m}}\right)\mathrm{d}z$$

$$=(1 + c_1\varepsilon)\sqrt{\frac{t-m}{m}}\int_{\mathbb{R}} H(\sqrt{\Psi''(\lambda_*)(t-m)}w)1_{\{\sqrt{\Psi''(\lambda_*)(t-m)}w\geq-\varepsilon\}}\phi\left((w-u)\sqrt{\frac{t-m}{m}}\right)dw.$$

Using integration by parts, we get that for any $x \in \mathbb{R}_+$,

$$A_1(x) \leq \int_{\mathbb{R}_+} \varphi'_t(u)\mathbf{P}_x^{\lambda_*}\left(\frac{\xi_{t-m}}{\sqrt{\Psi''(\lambda_*)(t-m)}} > u, \tau_0^- > t - m\right)du. \tag{3.25}$$

It follows from Lemma 3.3 that for $t - m > T_0(\varepsilon)$,

$$\left|\mathbf{P}_x^{\lambda_*}\left(\frac{\xi_{t-m}}{\sqrt{\Psi''(\lambda_*)(t-m)}} > u, \tau_0^- > t - m\right) - \frac{2R^*(x)}{\sqrt{2\pi(t-m)\Psi''(\lambda_*)}}\int_u^\infty \rho(z)dz\right| \leq \frac{C_4(\varepsilon)(1+x)}{(t-m)^{\frac{1}{2}+\varepsilon}},$$

which together with (3.25) implies that there exists a constant $c_5(\varepsilon)$ such that for $t - m > T_0(\varepsilon)$,

$$A_1(x) - \frac{2R^*(x)}{\sqrt{2\pi(t-m)\Psi''(\lambda_*)}}\int_{\mathbb{R}_+} \varphi'_t(u)e^{-\frac{u^2}{2}}du \leq \frac{c_5(\varepsilon)(1+x)}{(t-m)^{\frac{1}{2}+\varepsilon}}\int_{\mathbb{R}_+}|\varphi'_t(u)|du. \tag{3.26}$$

By the definition of $\varphi_t$ and a change of variables, we get that

$$\int_{\mathbb{R}_+}|\varphi'_t(u)|du \leq (1+c_1\varepsilon)\int_{\mathbb{R}_+}\int_{\mathbb{R}}\frac{t-m}{m}H(\sqrt{\Psi''(\lambda_*)(t-m)}w)1_{\{\sqrt{\Psi''(\lambda_*)(t-m)}w\geq-\varepsilon\}}\left|\phi'\left((w-u)\sqrt{\frac{t-m}{m}}\right)\right|dwdu$$

$$= (1+c_1\varepsilon)\int_{\mathbb{R}_+}\int_{\mathbb{R}} H(\sqrt{\Psi''(\lambda_*)m}u)1_{\{\sqrt{\Psi''(\lambda_*)m}u\geq-\varepsilon\}}|\phi'(u-y)|dudy$$

$$= (1+c_1\varepsilon)\int_{\mathbb{R}} H(\sqrt{\Psi''(\lambda_*)m}u)1_{\{\sqrt{\Psi''(\lambda_*)m}u\geq-\varepsilon\}}du\int_{\mathbb{R}_+}|\phi'(u-y)|dy.$$

Since there exists a constant $c_6 > 0$ such that $\int_{\mathbb{R}_+}|\phi'(u-y)|dy \leq c_6$, a change of variables yields that

$$\int_{\mathbb{R}_+}|\varphi'_t(u)|du \leq c_6(1+c_1\varepsilon)\frac{\|H1_{[-\varepsilon,\infty)}\|_1}{\sqrt{\Psi''(\lambda_*)m}}. \tag{3.27}$$

Using integration by parts, we get

$$\int_{\mathbb{R}_+}\varphi'_t(y)e^{-\frac{y^2}{2}}dy = \int_{\mathbb{R}_+}\varphi_t(y)\rho(y)dy$$

$$= (1+c_1\varepsilon)\int_{\mathbb{R}_+}\int_{\mathbb{R}}\sqrt{\frac{t-m}{m}}H(\sqrt{\Psi''(\lambda_*)(t-m)}w)1_{\{\sqrt{\Psi''(\lambda_*)(t-m)}w\geq-\varepsilon\}}\phi\left((w-y)\sqrt{\frac{t-m}{m}}\right)dw\rho(y)dy$$

$$= (1+c_1\varepsilon)\int_{\mathbb{R}_+}\int_{\mathbb{R}}\sqrt{\frac{t}{m}}H(\sqrt{\Psi''(\lambda_*)t}u)1_{\{\sqrt{\Psi''(\lambda_*)t}u\geq-\varepsilon\}}\phi\left((u-z)\sqrt{\frac{t}{m}}\right)du\rho\left(\sqrt{\frac{t}{t-m}}z\right)\sqrt{\frac{t}{t-m}}dz.$$

According to [16, Lemma 3.3], for any $v \in (0, \frac{1}{2}]$ and $s \geq 0$, it holds that

$$\sqrt{1-v}\rho(s) \leq \phi_v * \rho_{1-v}(s) \leq \sqrt{1-v}\rho(s) + \sqrt{v}e^{-\frac{s^2}{2v}}. \tag{3.28}$$

Letting $v = \frac{m}{t}$, we get

$$\int_{\mathbb{R}_+}\varphi'_t(y)e^{-\frac{y^2}{2}}dy = (1+c_1\varepsilon)\int_{\mathbb{R}} H(\sqrt{\Psi''(\lambda_*)t}u)1_{\{\sqrt{\Psi''(\lambda_*)t}u\geq-\varepsilon\}}\phi_{\frac{m}{t}} * \rho_{\frac{t-m}{t}}(u)du$$

$$= \frac{(1+c_1\varepsilon)}{\sqrt{\Psi''(\lambda_*)t}}\int_{-\varepsilon}^\infty H(w)\phi_{\frac{m}{t}} * \rho_{\frac{t-m}{t}}\left(\frac{w}{\sqrt{\Psi''(\lambda_*)t}}\right)dw$$

$$\leq \frac{(1+c_1\varepsilon)}{\sqrt{\Psi''(\lambda_*)t}}\int_{-\varepsilon}^\infty H(w)\left(\sqrt{\frac{t-m}{t}}\rho\left(\frac{w}{\sqrt{\Psi''(\lambda_*)t}}\right) + \sqrt{\frac{m}{t}}e^{-\frac{w^2}{2\Psi''(\lambda_*)t}}\right)dw.$$

Combining this with (3.22), (3.24), (3.26) (3.27), and using the fact that $\rho(z) = 0$ for $z \leq 0$ and noticing that $m = [\varepsilon t]$, we get that there exist positive constants $c_7$ (independent of $\varepsilon$) and $t_1(\varepsilon)$, $c_8(\varepsilon)$ such that for $t > t_1(\varepsilon)$,

$$\mathbf{E}_x^{\lambda_*}\left(h(\xi_t)1_{\{\tau_0^->t\}}\right) \leq \frac{2(1+c_7\varepsilon)R^*(x)}{\sqrt{2\pi}\Psi''(\lambda_*)t}\int_{\mathbb{R}_+} H(w)\rho\left(\frac{w}{\sqrt{\Psi''(\lambda_*)t}}\right)dw$$

$$+ c_7\sqrt{\varepsilon}\frac{2R^*(x)}{\Psi''(\lambda_*)t}\int_{-\varepsilon}^\infty H(w)\phi\left(\frac{w}{\sqrt{\Psi''(\lambda_*)t}}\right)dw + c_8(\varepsilon)(1+x)\|H1_{[-\varepsilon,\infty)}\|_1\left(\frac{1}{t^{1+\varepsilon}} + \frac{1}{t^{1+\delta/2}}\right).$$

The proof is complete. $\quad\square$

**Lemma 3.6.** *Assume that $\xi$ is a Lévy process satisfying* **(H1)**, **(H2)**, **(H3)** *and* $\mathbf{E}_0(\xi_1) < 0$. *Then one can find a constant $C_7 > 0$ with the property that for any $\varepsilon \in (0, \varepsilon_0)$ there exist positive constants $T_2(\varepsilon)$ and $C_8(\varepsilon)$ such that for any $x > 0$, $t > T_2(\varepsilon)$ and any Borel functions $h, H : \mathbb{R} \to \mathbb{R}_+$ satisfying $h \leq_\varepsilon H$ and $\int_{\mathbb{R}_+} H(z - \varepsilon)(1 + z)\mathrm{d}z < \infty$,*

$$\mathbf{E}_x^{\lambda_*}\left(h(\xi_t)1_{\{\tau_0^- > t\}}\right) \leq \left(1 + C_7 t^{-1/2} + C_7\sqrt{\varepsilon}\right)\frac{2R^*(x)}{\sqrt{2\pi\Psi''(\lambda_*)^3}t^{3/2}}\int_{\mathbb{R}_+} H(z - \varepsilon)\widehat{R}^*(z)\mathrm{d}z + \frac{C_8(\varepsilon)R^*(x)}{\sqrt{2\pi\Psi''(\lambda_*)^3}t^{3/2+\varepsilon}}\int_{\mathbb{R}_+} H(z - \varepsilon)(1 + z)\mathrm{d}z$$

$$+ \frac{C_8(\varepsilon)(1 + x)}{\sqrt{t}}\left(\frac{1}{t^{1+\varepsilon}} + \frac{1}{t^{1+\delta/2}}\right)\int_{\mathbb{R}_+} H(z - \varepsilon)(1 + z)\mathrm{d}z.$$

**Proof.** Fix $\varepsilon \in (0, \varepsilon_0)$ and let $h, H : \mathbb{R} \to \mathbb{R}_+$ satisfying $h \leq_\varepsilon H$. For any $z \in \mathbb{R}$, we define

$$H_m(z) := \mathbf{E}_0^{\lambda_*}\left(H(\xi_m + z)1_{\{\tau_{-z-\varepsilon}^- > m\}}\right) = \mathbf{E}_z^{\lambda_*}\left(H(\xi_m)1_{\{\tau_{-\varepsilon}^- > m\}}\right). \tag{3.29}$$

Fix $t \geq 2$ and set $m = [t/2]$. For any $y > 0$, we have

$$I_m(y) := \mathbf{E}_y^{\lambda_*}\left(h(\xi_m)1_{\{\tau_0^- > m\}}\right) \leq \mathbf{E}_y^{\lambda_*}\left(H(\xi_m + v)1_{\{\tau_{-v-\varepsilon}^- > m\}}\right) = H_m(y + v), \quad |v| \leq \varepsilon. \tag{3.30}$$

Consequently, $I_m \leq_\varepsilon H_m$. By the Markov property,

$$\mathbf{E}_x^{\lambda_*}\left(h(\xi_t)1_{\{\tau_0^- > t\}}\right) = \int_{\mathbb{R}_+} \mathbf{E}_y^{\lambda_*}\left(h(\xi_m)1_{\{\tau_0^- > m\}}\right)\mathbf{P}_x^{\lambda_*}\left(\xi_{t-m} \in \mathrm{d}y, \tau_0^- > t - m\right)$$

$$= \int_{\mathbb{R}_+} I_m(y)\mathbf{P}_x^{\lambda_*}\left(\xi_{t-m} \in \mathrm{d}y, \tau_0^- > t - m\right) = \mathbf{E}_x^{\lambda_*}\left(I_m(\xi_{t-m}), \tau_0^- > t - m\right).$$

Now applying Lemma 3.5 with $h = I_m$, we get that for $t - m > T_1(\varepsilon)$,

$$\mathbf{E}_x^{\lambda_*}\left(h(\xi_t)1_{\{\tau_0^- > t\}}\right) \leq J_1 + J_2 + J_3, \tag{3.31}$$

where

$$J_1 := \frac{2(1 + C_5\varepsilon)R^*(x)}{\sqrt{2\pi}\Psi''(\lambda_*)(t-m)}\int_{\mathbb{R}_+} H_m(w)\rho\left(\frac{w}{\sqrt{\Psi''(\lambda_*)(t-m)}}\right)\mathrm{d}w,$$

$$J_2 := C_5\sqrt{\varepsilon}\frac{2R^*(x)}{\Psi''(\lambda_*)(t-m)}\int_{-\varepsilon}^{\infty} H_m(w)\phi\left(\frac{w}{\sqrt{\Psi''(\lambda_*)(t-m)}}\right)\mathrm{d}w,$$

$$J_3 := C_6(\varepsilon)\left(\frac{1}{(t-m)^{1+\varepsilon}} + \frac{1}{(t-m)^{1+\delta/2}}\right)(1 + x)\|H_m 1_{[-\varepsilon, \infty)}\|_1.$$

We will deal with the upper bounds of $J_i$ separately. We first deal with $J_1$. Note that

$$J_1 = \frac{2(1 + C_5\varepsilon)R^*(x)}{\sqrt{2\pi}\Psi''(\lambda_*)(t-m)}\int_{\mathbb{R}_+} \mathbf{E}_w^{\lambda_*}\left(H(\xi_m)1_{\{\tau_{-\varepsilon}^- > m\}}\right)\rho\left(\frac{w}{\sqrt{\Psi''(\lambda_*)(t-m)}}\right)\mathrm{d}w$$

$$= \frac{2(1 + C_5\varepsilon)R^*(x)}{\sqrt{2\pi}\Psi''(\lambda_*)(t-m)}\int_{\mathbb{R}} \mathbf{E}_{w+\varepsilon}^{\lambda_*}\left(H(\xi_m - \varepsilon)1_{\{\tau_0^- > m\}}\right)1_{\{w+\varepsilon \geq 0\}}\rho\left(\frac{w}{\sqrt{\Psi''(\lambda_*)(t-m)}}\right)\mathrm{d}w$$

$$= \frac{2(1 + C_5\varepsilon)R^*(x)}{\sqrt{2\pi}\Psi''(\lambda_*)(t-m)}\int_{\mathbb{R}_+} \mathbf{E}_w^{\lambda_*}\left(H(\xi_m - \varepsilon)1_{\{\tau_0^- > m\}}\right)\rho\left(\frac{w - \varepsilon}{\sqrt{\Psi''(\lambda_*)(t-m)}}\right)\mathrm{d}w$$

$$= \frac{2(1 + C_5\varepsilon)R^*(x)}{\sqrt{2\pi}\Psi''(\lambda_*)(t-m)}\int_{\mathbb{R}_+} H(w - \varepsilon)\mathbf{E}_w^{\lambda_*}\left(\rho\left(\frac{\widehat{\xi}_m - \varepsilon}{\sqrt{\Psi''(\lambda_*)(t-m)}}\right)1_{\{\widehat{\tau}_0^- > m\}}\right)\mathrm{d}w,$$

where in the last equality we used (3.15). Using integration by parts, we get for any $z \in \mathbb{R}_+$,

$$\mathbf{E}_z^{\lambda_*}\left(\rho\left(\frac{\widehat{\xi}_m - \varepsilon}{\sqrt{\Psi(\lambda_*)(t-m)}}\right)1_{\{\widehat{\tau}_0^- > m\}}\right) = \int_{\mathbb{R}_+} \rho'(u)\mathbf{P}_z^{\lambda_*}\left(\frac{\widehat{\xi}_m - \varepsilon}{\sqrt{\Psi(\lambda_*)(t-m)}} > u, \widehat{\tau}_0^- > m\right)\mathrm{d}u$$

$$= \int_{\mathbb{R}_+} \rho'(u)\mathbf{P}_z^{\lambda_*}\left(\frac{\widehat{\xi}_m}{\sqrt{\Psi''(\lambda_*)m}} > \frac{u\sqrt{\Psi(\lambda_*)(t-m)} + \varepsilon}{\sqrt{\Psi''(\lambda_*)m}}, \widehat{\tau}_0^- > m\right)\mathrm{d}u. \tag{3.32}$$

Set

$$u_{m,\varepsilon} := \frac{u\sqrt{\Psi(\lambda_*)(t-m)} + \varepsilon}{\sqrt{\Psi''(\lambda_*)m}}.$$

Applying Lemma 3.3 to $\widehat{\xi}$, we get that for $m > T_0(\varepsilon)$,

$$\left|\mathbf{P}_z^{\lambda_*}\left(\frac{\widehat{\xi}_m}{\sqrt{\Psi''(\lambda_*)m}} > u_{m,\varepsilon}, \widehat{\tau}_0^- > m\right) - \frac{2\widehat{R}^*(z)}{\sqrt{2\pi m\Psi''(\lambda_*)}}\int_{u_{m,\varepsilon}}^{\infty} \rho(y)\mathrm{d}y\right| \leq \frac{C_4(\varepsilon)(1 + z)}{m^{1/2+\varepsilon}}.$$

Substituting this into (3.32) and using the fact that $\int_{\mathbb{R}_+} \rho'(u)\mathrm{d}u \leq c_1$ for some $c_1 > 0$, we get that

$$\left| \mathbf{E}_z^{\lambda_*}\left( \rho\left( \frac{\widehat{\xi}_m - \varepsilon}{\sqrt{\Psi''(\lambda_*)(t-m)}} \right) 1_{\{\widehat{\tau}_0^- > m\}} \right) - \frac{2\widehat{R}^*(z)}{\sqrt{2\pi m \Psi''(\lambda_*)}} \int_{\mathbb{R}_+} \rho'(u) e^{-\frac{u_{m,\varepsilon}^2}{2}} \mathrm{d}u \right| \leq \frac{C_4(\varepsilon)(1+z)}{m^{1/2+\varepsilon}} \int_{\mathbb{R}_+} \rho'(u)\mathrm{d}u \leq c_1 \frac{C_4(\varepsilon)(1+z)}{m^{1/2+\varepsilon}}. \tag{3.33}$$

Using integration by parts again and the boundedness of $\rho'$, we get that there exists a positive constant $c_2$ (independent of $\varepsilon$) such that

$$\int_{\mathbb{R}_+} \rho'(u) e^{-\frac{u_{m,\varepsilon}^2}{2}} \mathrm{d}u = \sqrt{\frac{t-m}{m}} \int_{\mathbb{R}_+} \rho(u)\rho(u_{m,\varepsilon})\mathrm{d}u \leq \sqrt{\frac{t-m}{m}} \int_{\mathbb{R}_+} \rho(u)\rho\left( \frac{u\sqrt{t-m}}{\sqrt{m}} \right)\mathrm{d}u + \frac{c_2 \varepsilon}{\sqrt{t}}, \tag{3.34}$$

where in the last inequality we used the mean value theorem. By a change of variables, we see that

$$\sqrt{\frac{t-m}{m}} \int_{\mathbb{R}_+} \rho(u)\rho\left( \frac{u\sqrt{t-m}}{\sqrt{m}} \right)\mathrm{d}u = \frac{1}{\sqrt{m}} \int_{\mathbb{R}_+} \rho\left( \frac{y}{\sqrt{t-m}} \right)\rho\left( \frac{y}{\sqrt{m}} \right)\mathrm{d}y$$

$$= \frac{1}{\sqrt{m}} \int_{\mathbb{R}_+} \frac{y^2}{\sqrt{(t-m)m}} e^{-\frac{ty^2}{2m(t-m)}} \mathrm{d}y = \frac{\sqrt{m}(t-m)}{t^{3/2}} \int_{\mathbb{R}_+} y^2 e^{-\frac{y^2}{2}} \mathrm{d}y = \frac{\sqrt{2\pi m}(t-m)}{2t^{3/2}}. \tag{3.35}$$

Combining this with (3.33), (3.34) and (3.35), we get that there exist positive constants $c_4$ (independent of $\varepsilon$) and $c_5(\varepsilon)$ such that

$$J_1 \leq \left( 1 + c_4 t^{-\frac{1}{2}} \right) \frac{2(1+c_4\varepsilon)R^*(x)}{\sqrt{2\pi \Psi''(\lambda_*)^3} t^{3/2}} \int_{\mathbb{R}_+} H(z-\varepsilon)\widehat{R}^*(z)\mathrm{d}z + \frac{c_5(\varepsilon)R^*(x)}{\sqrt{2\pi \Psi''(\lambda_*)^3} t^{3/2+\varepsilon}} \int_{\mathbb{R}_+} H(z-\varepsilon)(1+z)\mathrm{d}z. \tag{3.36}$$

Next, we deal with $J_2$. Note that

$$J_2 = \frac{2C_5\sqrt{\varepsilon}R^*(x)}{\Psi''(\lambda_*)(t-m)} \int_{-\varepsilon}^{\infty} \mathbf{E}_w^{\lambda_*}\left( H(\xi_m)1_{\{\tau_{-\varepsilon}^- > m\}} \right)\phi\left( \frac{w}{\sqrt{\Psi''(\lambda_*)(t-m)}} \right)\mathrm{d}w$$

$$= \frac{2C_5\sqrt{\varepsilon}R^*(x)}{\Psi''(\lambda_*)(t-m)} \int_{-\varepsilon}^{\infty} \mathbf{E}_{w+\varepsilon}^{\lambda_*}\left( H(\xi_m - \varepsilon)1_{\{\tau_0^- > m\}} \right)\phi\left( \frac{w}{\sqrt{\Psi''(\lambda_*)(t-m)}} \right)\mathrm{d}w$$

$$= \frac{2C_5\sqrt{\varepsilon}R^*(x)}{\Psi''(\lambda_*)(t-m)} \int_{\mathbb{R}_+} \mathbf{E}_w^{\lambda_*}\left( H(\xi_m - \varepsilon)1_{\{\tau_0^- > m\}} \right)\phi\left( \frac{w-\varepsilon}{\sqrt{\Psi''(\lambda_*)(t-m)}} \right)\mathrm{d}w$$

$$= \frac{2C_5\sqrt{\varepsilon}R^*(x)}{\Psi''(\lambda_*)(t-m)} \int_{\mathbb{R}_+} H(w-\varepsilon)\mathbf{E}_w^{\lambda_*}\left( \phi\left( \frac{\widehat{\xi}_m - \varepsilon}{\sqrt{\Psi''(\lambda_*)(t-m)}} \right) 1_{\{\widehat{\tau}_0^- > m\}} \right)\mathrm{d}w,$$

where in the last equality we used (3.15). Now repeating the argument leading to (3.36), we get that there exist positive constants $c_6$ (independent of $\varepsilon$) and $c_7(\varepsilon)$ such that

$$J_2 \leq \left( 1 + c_6\varepsilon t^{-\frac{1}{2}} \right) c_6\sqrt{\varepsilon} \frac{2R^*(x)}{\sqrt{2\pi \Psi''(\lambda_*)^3} t^{3/2}} \int_{\mathbb{R}_+} H(z-\varepsilon)\widehat{R}^*(z)\mathrm{d}z + \frac{c_7(\varepsilon)R^*(x)}{\sqrt{2\pi \Psi''(\lambda_*)^3} t^{3/2+\varepsilon}} \int_{\mathbb{R}_+} H(z-\varepsilon)(1+z)\mathrm{d}z. \tag{3.37}$$

Finally, we deal with $J_3$. By the definition of $H_m$ and (3.23), we have

$$\|H_m 1_{[-\varepsilon,\infty)}\|_1 = \int_{\mathbb{R}} \mathbf{E}_y^{\lambda_*}\left( H(\xi_m)1_{\{\tau_{-\varepsilon}^- > m\}} \right)1_{\{y \geq -\varepsilon\}}\mathrm{d}y = \int_{\mathbb{R}} \mathbf{E}_{y+\varepsilon}^{\lambda_*}\left( H(\xi_m - \varepsilon)1_{\{\tau_0^- > m\}} \right)1_{\{y \geq -\varepsilon\}}\mathrm{d}y$$

$$= \int_{\mathbb{R}_+} \mathbf{E}_y^{\lambda_*}\left( H(\xi_m - \varepsilon)1_{\{\tau_0^- > m\}} \right)\mathrm{d}y = \int_{\mathbb{R}_+} H(z-\varepsilon)\mathbf{P}_z(\widehat{\tau}_0^- > m)\mathrm{d}z \leq c_8 \int_{\mathbb{R}_+} H(z-\varepsilon)\frac{1+z}{\sqrt{m}}\mathrm{d}z, \tag{3.38}$$

where in the last equality we used (3.15) and $c_8$ is a positive constant independent of $\varepsilon$. Since $m = [t/2]$, there exists a positive constant $c_9(\varepsilon)$ such that

$$J_3 \leq \frac{c_9(\varepsilon)(1+x)}{\sqrt{t}} \left( \frac{1}{t^{1+\varepsilon}} + \frac{1}{t^{1+\delta/2}} \right) \int_{\mathbb{R}_+} (1+z)H(z-\varepsilon)\mathrm{d}z. \tag{3.39}$$

Combining (3.31), (3.36), (3.37) and (3.39), we complete the proof. $\quad\square$

**Lemma 3.7.** *Assume that $\xi$ is a Lévy process satisfying* **(H1)**, **(H2)**, **(H3)** *and* $\mathbf{E}_0(\xi_1) < 0$. *Then one can find positive constants $C_9$ and $q$ with the property that for any $\varepsilon \in (0, \varepsilon_0)$ there exist positive constants $T_3(\varepsilon)$ and $C_{10}(\varepsilon)$ such that for any $x > 0$, $t > T_3(\varepsilon)$ and any integrable functions $h, H, g : \mathbb{R} \to \mathbb{R}_+$ satisfying $g \leq_\varepsilon h \leq_\varepsilon H$,*

$$\mathbf{E}_x^{\lambda_*}\left( h(\xi_t)1_{\{\tau_0^- > t\}} \right) \geq \frac{2R^*(x)}{\sqrt{2\pi \Psi''(\lambda_*)}t} \int_{\mathbb{R}_+} \left( g(w)1_{\{w \geq \varepsilon\}} - C_9\varepsilon h(w) \right)\rho\left( \frac{w}{\sqrt{\Psi''(\lambda_*)t}} \right)\mathrm{d}w$$

$$- C_9\varepsilon^{1/12} \frac{2R^*(x)}{\sqrt{2\pi \Psi''(\lambda_*)}t} \int_{-\varepsilon}^{\infty} H(u)\phi\left( \frac{w}{\sqrt{\Psi''(\lambda_*)t}} \right)\mathrm{d}w - C_{10}(\varepsilon)(1+x)\|H 1_{[-\varepsilon,\infty)}\|_1 \left( \frac{1}{t^{1+\varepsilon}} + \frac{1}{t^{1+\delta/2}} + \frac{1}{t^{1+q}} \right),$$

*where $\Psi$ is the Laplace exponent of $\xi$.*

**Proof.** Fix $\epsilon \in (0, \epsilon_0)$ and let $h, H, g : \mathbb{R} \to \mathbb{R}_+$ be integrable functions satisfying $g \leq_\epsilon h \leq_\epsilon H$. Then $g 1_{[\epsilon,\infty)} \leq_\epsilon h 1_{[0,\infty)} \leq_\epsilon H 1_{[-\epsilon,\infty)}$. Fix $t \geq 1$ and set $m = [\epsilon t]$. By the Markov property,

$$
\mathbf{E}_x^{\lambda_*}\left( h(\xi_t) 1_{\{\tau_0^- > t\}} \right) = \int_{\mathbb{R}_+} \mathbf{E}_y^{\lambda_*}\left( h(\xi_m) 1_{\{\tau_0^- > m\}} \right) \mathbf{P}_x^{\lambda_*}\left( \xi_{t-m} \in dy, \tau_0^- > t - m \right)
$$

$$
= \int_{\mathbb{R}_+} \mathbf{E}_y^{\lambda_*}\left( h(\xi_m) 1_{\{\xi_m \geq 0\}} \right) \mathbf{P}_x^{\lambda_*}\left( \xi_{t-m} \in dy, \tau_0^- > t - m \right) - \int_{\mathbb{R}_+} \mathbf{E}_y^{\lambda_*}\left( h(\xi_m) 1_{\{\xi_m \geq 0\}} 1_{\{\tau_0^- \leq m\}} \right) \mathbf{P}_x^{\lambda_*}\left( \xi_{t-m} \in dy, \tau_0^- > t - m \right)
$$

$$
=: I_1(t) - I_2(t) =: I_1(t) - I_2^1(t) - I_2^2(t),
$$

where

$$
I_1(t) := \int_{\mathbb{R}_+} \mathbf{E}_y^{\lambda_*}\left( h(\xi_m) 1_{\{\xi_m \geq 0\}} \right) \mathbf{P}_x^{\lambda_*}\left( \xi_{t-m} \in dy, \tau_0^- > t - m \right),
$$

$$
I_2^1(t) := \int_0^{\epsilon^{1/6}\sqrt{[t]}} \mathbf{E}_y^{\lambda_*}\left( h(\xi_m) 1_{\{\xi_m \geq 0\}} 1_{\{\tau_0^- \leq m\}} \right) \mathbf{P}_x^{\lambda_*}\left( \xi_{t-m} \in dy, \tau_0^- > t - m \right),
$$

$$
I_2^2(t) := \int_{\epsilon^{1/6}\sqrt{[t]}}^\infty \mathbf{E}_y^{\lambda_*}\left( h(\xi_m) 1_{\{\xi_m \geq 0\}} 1_{\{\tau_0^- \leq m\}} \right) \mathbf{P}_x^{\lambda_*}\left( \xi_{t-m} \in dy, \tau_0^- > t - m \right).
$$

The proof of the lemma is divided into the following three steps.

**Step 1.** In this step, we give a lower bound for $I_1(t)$. By [11, Theorem 2.7], there exist positive constants $c_1$ (independent of $\epsilon$) and $c_2(\epsilon)$ such that for any $m \geq 1$,

$$
\mathbf{E}_y^{\lambda_*}\left( h(\xi_m) 1_{\{\xi_m \geq 0\}} \right) \geq \frac{1}{\sqrt{\Psi''(\lambda_*)m}} \int_{\mathbb{R}} \left( g(z) 1_{\{z \geq \epsilon\}} - c_1 \epsilon h(z) 1_{\{z \geq 0\}} \right) \phi\left( \frac{z - y}{\sqrt{\Psi''(\lambda_*)m}} \right) dz - \frac{c_2(\epsilon)}{m^{(1+\delta)/2}} \| h 1_{[0,\infty)} \|_1.
$$

Note that $\| h 1_{[0,\infty)} \|_1 \leq \| H 1_{[0,\infty)} \|_1$. Following the analysis of $A_1(x)$ in Lemma 3.5 and using the lower bound in (3.28), we see that there exist positive constants $t_1(\epsilon)$ and $c_3(\epsilon)$ such that for $t > t_1(\epsilon)$,

$$
\frac{1}{\sqrt{\Psi''(\lambda_*)m}} \int_{\mathbb{R}_+} \left( \int_{\mathbb{R}} g(z) 1_{\{z \geq \epsilon\}} \phi\left( \frac{z - y}{\sqrt{\Psi''(\lambda_*)m}} \right) dz \right) \mathbf{P}_x^{\lambda_*}\left( \xi_{t-m} \in dy, \tau_0^- > t - m \right)
$$

$$
\geq \frac{2R^*(x)}{\sqrt{2\pi}\Psi''(\lambda_*)t} \int_{\mathbb{R}_+} g(w) 1_{\{w \geq \epsilon\}} \rho\left( \frac{w}{\sqrt{\Psi''(\lambda_*)t}} \right) dw - \frac{c_3(\epsilon)(1 + x) \| H 1_{[0,\infty)} \|_1}{t^{1+\epsilon}}.
$$

and using the upper bound in (3.28), we have

$$
\frac{c_1 \epsilon}{\sqrt{\Psi''(\lambda_*)m}} \int_{\mathbb{R}_+} \left( \int_{\mathbb{R}} h(z) 1_{\{z \geq 0\}} \phi\left( \frac{z - y}{\sqrt{\Psi''(\lambda_*)m}} \right) dz \right) \mathbf{P}_x^{\lambda_*}\left( \xi_{t-m} \in dy, \tau_0^- > t - m \right)
$$

$$
\leq \frac{2 c_4 \epsilon R^*(x)}{\sqrt{2\pi}\Psi''(\lambda_*)t} \int_{\mathbb{R}_+} h(w) \rho\left( \frac{w}{\sqrt{\Psi''(\lambda_*)t}} \right) dw + \frac{2 c_4 \sqrt{\epsilon} R^*(x)}{\Psi''(\lambda_*)t} \int_{\mathbb{R}_+} h(w) \phi\left( \frac{w}{\sqrt{\Psi''(\lambda_*)t}} \right) dw + \frac{c_3(\epsilon)(1 + x) \| H 1_{[0,\infty)} \|_1}{t^{1+\epsilon}},
$$

where $c_4$ is a positive constant independent of $\epsilon$. Thus there exists a positive constant $c_5(\epsilon)$ such that for $t > t_1(\epsilon)$,

$$
I_1(t) \geq \frac{2R^*(x)}{\sqrt{2\pi}\Psi''(\lambda_*)t} \int_{\mathbb{R}_+} \left( g(w) 1_{\{w \geq \epsilon\}} - c_4 \epsilon h(w) \right) \rho\left( \frac{w}{\sqrt{\Psi''(\lambda_*)t}} \right) dw - c_4 \sqrt{\epsilon} \frac{2R^*(x)}{\Psi''(\lambda_*)t} \int_{\mathbb{R}_+} h(w) \phi\left( \frac{w}{\sqrt{\Psi''(\lambda_*)t}} \right) dw
$$

$$
- c_5(\epsilon)(1 + x) \| H 1_{[0,\infty)} \|_1 \left( \frac{1}{t^{1+\epsilon}} + \frac{1}{t^{1+\delta/2}} \right). \tag{3.40}
$$

**Step 2.** Next, we give an upper bound for $I_2^1(t)$. Combining (3.21) and (3.23), we get that there exist positive constants $c_6$ (independent of $\epsilon$) and $c_7(\epsilon)$ such that

$$
I_2^1(t) \leq \int_0^{\epsilon^{1/6}\sqrt{[t]}} \mathbf{E}_y^{\lambda_*}\left( h(\xi_m) 1_{\{\xi_m \geq 0\}} \right) \mathbf{P}_x^{\lambda_*}\left( \xi_{t-m} \in dy, \tau_0^- > t - m \right)
$$

$$
\leq \int_0^{\epsilon^{1/6}\sqrt{[t]}} \frac{1 + c_6 \epsilon}{\sqrt{\Psi''(\lambda_*)m}} \int_{\mathbb{R}} H(z) 1_{\{z \geq -\epsilon\}} \phi\left( \frac{z - y}{\sqrt{\Psi''(\lambda_*)m}} \right) dz \mathbf{P}_x^{\lambda_*}\left( \xi_{t-m} \in dy, \tau_0^- > t - m \right) + \frac{c_7(\epsilon)(1 + x) \| H 1_{[-\epsilon,\infty)} \|_1}{m^{(1+\delta)/2}\sqrt{t - m}}.
$$

For any $u \in \mathbb{R}$, define

$$
J(u) := \int_0^{\epsilon^{1/6}\sqrt{[t]}} \left( \frac{1}{\sqrt{\Psi''(\lambda_*)m}} \phi\left( \frac{u - y}{\sqrt{\Psi''(\lambda_*)m}} \right) \right) \mathbf{P}_x^{\lambda_*}\left( \xi_{t-m} \in dy, \tau_0^- > t - m \right).
$$

Then by Fubini's theorem, we have

$$
\int_0^{\epsilon^{1/6}\sqrt{[t]}} \frac{1}{\sqrt{\Psi''(\lambda_*)m}} \int_{\mathbb{R}} H(z) 1_{\{z \geq -\epsilon\}} \phi\left( \frac{z - y}{\sqrt{\Psi''(\lambda_*)m}} \right) dz \mathbf{P}_x^{\lambda_*}\left( \xi_{t-m} \in dy, \tau_0^- > t - m \right) = \int_{\mathbb{R}} H(u) 1_{\{u \geq -\epsilon\}} J(u) du.
$$

For any $u \in \mathbb{R}_+$, define

$$F_u(y) := \frac{1}{\sqrt{\Psi''(\lambda_*)m}} \phi\left(\frac{u-y}{\sqrt{\Psi''(\lambda_*)m}}\right).$$

Using the definition of $J(u)$ and integration by parts, we get

$$
\begin{aligned}
J(u) &= \int_0^\infty F_u(y) \mathbf{P}_x^{\lambda_*}\left(\xi_{t-m} \in \mathrm{d}y, \xi_{t-m} \le \varepsilon^{1/6}\sqrt{[t]}, \tau_0^- > t-m\right) \\
&\le \int_0^\infty F_u'(y) \mathbf{P}_x^{\lambda_*}\left(\xi_{t-m} > y, \xi_{t-m} \in (0, \varepsilon^{1/6}\sqrt{[t]}], \tau_0^- > t-m\right)\mathrm{d}y \\
&= \int_0^{\varepsilon^{1/6}\sqrt{[t]}} F_u'(y) \mathbf{P}_x^{\lambda_*}\left(\xi_{t-m} \in (y, \varepsilon^{1/6}\sqrt{[t]}], \tau_0^- > t-m\right)\mathrm{d}y.
\end{aligned}
$$

Since $m = [\varepsilon t]$, using Lemma 3.3, it holds that for $t - m > T_0(\varepsilon)$,

$$\left| \mathbf{P}_x^{\lambda_*}\left(\xi_{t-m} \in (y, \varepsilon^{1/6}\sqrt{[t]}], \tau_0^- > t-m\right) - \frac{2R^*(x)}{\sqrt{2\pi(t-m)\Psi''(\lambda_*)}} \int_{\frac{y}{\sqrt{\Psi''(\lambda_*)(t-m)}}}^{\frac{\varepsilon^{1/6}\sqrt{[t]}}{\sqrt{\Psi''(\lambda_*)(t-m)}}} \rho(z)\mathrm{d}z \right| \le \frac{C_4(\varepsilon)(1+x)}{(t-m)^{1/2+\varepsilon}}.$$

Now, using the fact

$$F_u(\varepsilon^{1/6}\sqrt{[t]}) - F_u(0) \le \frac{c_8(\varepsilon)}{\sqrt{t}},$$

for some $c_8(\varepsilon) > 0$, we get that there exists a positive constant $c_9(\varepsilon)$ such that for $t - m > T_0(\varepsilon)$,

$$J(u) \le \frac{c_9(\varepsilon)(1+x)}{(t-m)^{1/2+\varepsilon}} \frac{1}{\sqrt{t}} + \frac{2R^*(x)}{\sqrt{2\pi(t-m)\Psi''(\lambda_*)}} \int_0^{\varepsilon^{1/6}\sqrt{[t]}} F_u'(y)\left(\mathcal{R}\left(\frac{\varepsilon^{1/6}\sqrt{[t]}}{\sqrt{\Psi''(\lambda_*)(t-m)}}\right) - \mathcal{R}\left(\frac{y}{\sqrt{\Psi''(\lambda_*)(t-m)}}\right)\right)\mathrm{d}y.$$

Using integration by parts and the fact $F_u(0) \ge 0$ (see [11, (3.33)]), we get that there exists a constant $c_{10}$ (independent of $\varepsilon$) such that

$$\int_0^{\varepsilon^{1/6}\sqrt{[t]}} F_u'(y)\left(\mathcal{R}\left(\frac{\varepsilon^{1/6}\sqrt{[t]}}{\sqrt{\Psi''(\lambda_*)(t-m)}}\right) - \mathcal{R}\left(\frac{y}{\sqrt{\Psi''(\lambda_*)(t-m)}}\right)\right)\mathrm{d}y \le \frac{c_{10}\varepsilon^{1/12}}{\sqrt{t-m}} \phi\left(\frac{u}{\sqrt{\Psi''(\lambda_*)t}}\right).$$

It follows that there exist positive constants $c_{11}$ (independent of $\varepsilon$), and $t_2(\varepsilon)$, $c_{12}(\varepsilon)$ such that for $t > t_2(\varepsilon)$,

$$I_2^1(t) \le c_{11}\varepsilon^{1/12} \frac{2R^*(x)}{\sqrt{2\pi\Psi''(\lambda_*)t}} \int_{-\varepsilon}^\infty H(u)\phi\left(\frac{u}{\sqrt{\Psi''(\lambda_*)t}}\right)\mathrm{d}u + c_{12}(\varepsilon)(1+x)\|H 1_{[-\varepsilon,\infty)}\|_1 \left(\frac{1}{t^{1+\varepsilon}} + \frac{1}{t^{1+\delta/2}}\right). \tag{3.41}$$

**Step 3.** Finally, we study the upper bound for $I_2^2(t)$. By the definition of $I_2^2(t)$, we have

$$I_2^2(t) = \int_{\mathbb{R}} J_m(y) \mathbf{P}_x^{\lambda_*}\left(\xi_{t-m} \in \mathrm{d}y, \tau_0^- > t-m\right),$$

where $J_m(y) := \mathbf{E}_y^{\lambda_*}\left(h(\xi_m) 1_{\{\xi_m \ge 0\}} 1_{\{\tau_0^- \le m\}}\right) 1_{\{y > \varepsilon^{1/6}\sqrt{[t]}\}}$. For any $z \in \mathbb{R}$, define

$$M_m(z) := \mathbf{E}_z^{\lambda_*}\left(H(\xi_m) 1_{\{\xi_m \ge -\varepsilon\}} 1_{\{\tau_\varepsilon^- \le m\}}\right) 1_{\{z+\varepsilon > \varepsilon^{1/6}\sqrt{[t]}\}}.$$

Consequently, $J_m \le_\varepsilon M_m$. Applying Lemma 3.5 with $h$ and $H$ instead of $J_m$ and $M_m$, we get that for $t - m > T_1(\varepsilon)$,

$$
\begin{aligned}
I_2^2(t) \le{} & \frac{2(1+C_5\varepsilon)R^*(x)}{\sqrt{2\pi}\Psi''(\lambda_*)(t-m)} \int_{\mathbb{R}_+} M_m(w)\rho\left(\frac{w}{\sqrt{\Psi''(\lambda_*)(t-m)}}\right)\mathrm{d}w + \frac{2C_5\sqrt{\varepsilon}R^*(x)}{\sqrt{2\pi}\Psi''(\lambda_*)(t-m)} \int_{-\varepsilon}^\infty M_m(w)e^{-\frac{w^2}{2\Psi''(\lambda_*)t}}\mathrm{d}w \\
& + C_6(\varepsilon)(1+x)\|M_m 1_{[-\varepsilon,\infty)}\|_1 \left(\frac{1}{(t-m)^{1+\varepsilon}} + \frac{1}{(t-m)^{1+\delta/2}}\right).
\end{aligned}
\tag{3.42}
$$

Now we bound the three terms on the right-hand side of (3.42) from above. Using (3.17),

$$
\begin{aligned}
\|M_m 1_{[-\varepsilon,\infty)}\|_1 &= \int_{\mathbb{R}} \mathbf{E}_z^{\lambda_*}\left(H(\xi_m) 1_{\{\xi_m \ge -\varepsilon\}} 1_{\{\tau_\varepsilon^- \le m\}}\right) 1_{\{z+\varepsilon > \varepsilon^{1/6}\sqrt{[t]}\}} 1_{\{z \ge -\varepsilon\}}\mathrm{d}z \tag{3.43} \\
&\le \int_{\mathbb{R}} \mathbf{E}_z^{\lambda_*}\left(H(\xi_m) 1_{\{\xi_m \ge -\varepsilon\}}\right) 1_{\{z+\varepsilon > \varepsilon^{1/6}\sqrt{[t]}\}} 1_{\{z \ge -\varepsilon\}}\mathrm{d}z \\
&= \int_{\mathbb{R}} H(z) 1_{\{z \ge -\varepsilon\}} \mathbf{P}_z^{\lambda_*}\left(\widehat{\xi}_m + \varepsilon > \varepsilon^{1/6}\sqrt{[t]}, \widehat{\xi}_m \ge -\varepsilon\right)\mathrm{d}z \\
&\le \int_{-\varepsilon}^\infty H(z)\mathrm{d}z \le \|H 1_{[-\varepsilon,\infty)}\|_1.
\end{aligned}
$$

Moreover, using the definition of $M_m$ and (3.16), we get that

$$\int_{\mathbb{R}_+} M_m(w)\rho\left(\frac{w}{\sqrt{\Psi''(\lambda_*)(t-m)}}\right)dw$$

$$= \int_{\mathbb{R}_+} \mathbf{E}_w^{\lambda_*}\left(H(\xi_m)1_{\{\xi_m\geq-\varepsilon\}}1_{\{\tau_\varepsilon^-\leq m\}}\right)1_{\{w+\varepsilon>\varepsilon^{1/6}\sqrt{[t]}\}}\rho\left(\frac{w}{\sqrt{\Psi''(\lambda_*)(t-m)}}\right)dw$$

$$= \int_{\mathbb{R}} \mathbf{E}_{w+\varepsilon}^{\lambda_*}\left(H(\xi_m)1_{\{\xi_m\geq-\varepsilon\}}1_{\{\tau_\varepsilon^-\leq m\}}\right)1_{\{w+2\varepsilon>\varepsilon^{1/6}\sqrt{[t]}\}}\rho\left(\frac{w+\varepsilon}{\sqrt{\Psi''(\lambda_*)(t-m)}}\right)dw$$

$$= \int_{\mathbb{R}} \mathbf{E}_w^{\lambda_*}\left(H(\xi_m+\varepsilon)1_{\{\xi_m+\varepsilon\geq-\varepsilon\}}1_{\{\tau_0^-\leq m\}}\right)1_{\{w+2\varepsilon>\varepsilon^{1/6}\sqrt{[t]}\}}\rho\left(\frac{w+\varepsilon}{\sqrt{\Psi''(\lambda_*)(t-m)}}\right)dw$$

$$= \int_{\mathbb{R}} H(w+\varepsilon)1_{\{w\geq-2\varepsilon\}}\mathbf{E}_w^{\lambda_*}\left(\rho\left(\frac{\hat{\xi}_m+\varepsilon}{\sqrt{\Psi''(\lambda_*)(t-m)}}\right)1_{\{\hat{\xi}_m+2\varepsilon>\varepsilon^{1/6}\sqrt{[t]},\hat{\tau}_0^-\leq m\}}\right)dw$$

$$=: J_1(t) + J_2(t),$$

where

$$J_1(t) := \int_{-2\varepsilon}^{\varepsilon^{1/4}\sqrt{[t]}} H(w+\varepsilon)\mathbf{E}_w^{\lambda_*}\left(\rho\left(\frac{\hat{\xi}_m+\varepsilon}{\sqrt{\Psi''(\lambda_*)(t-m)}}\right)1_{\{\hat{\xi}_m+2\varepsilon>\varepsilon^{1/6}\sqrt{[t]},\hat{\tau}_0^-\leq m\}}\right)dw,$$

$$J_2(t) := \int_{\varepsilon^{1/4}\sqrt{[t]}}^{\infty} H(w+\varepsilon)\mathbf{E}_w^{\lambda_*}\left(\rho\left(\frac{\hat{\xi}_m+\varepsilon}{\sqrt{\Psi''(\lambda_*)(t-m)}}\right)1_{\{\hat{\xi}_m+2\varepsilon>\varepsilon^{1/6}\sqrt{[t]},\hat{\tau}_0^-\leq m\}}\right)dw.$$

Next, we consider the upper bounds of $J_1(t)$ and $J_2(t)$ separately. We claim that there exist positive constants $c_{13}$ and $q$ (both independent of $\varepsilon$), and $c_{14}(\varepsilon)$ such that

$$J_1(t) \leq c_{13}\varepsilon^{1/6}\int_{-2\varepsilon}^{\varepsilon^{1/4}\sqrt{[t]}} H(w+\varepsilon)dw, \tag{3.44}$$

and

$$J_2(t) \leq c_{13}\varepsilon^{1/12}\int_{\varepsilon^{1/4}\sqrt{[t]}}^{\infty} H(w+\varepsilon)\phi\left(\frac{w}{\sqrt{\Psi''(\lambda_*)t}}\right)dw + \frac{c_{14}(\varepsilon)}{t^q}\|H1_{[-\varepsilon,\infty)}\|_1. \tag{3.45}$$

Using (3.44), (3.45) and the fact that $\phi$ is bounded, we immediately get there exists a positive constant $c_{15}$ (independent of $\varepsilon$) such that

$$\int_{\mathbb{R}_+} M_m(w)\rho\left(\frac{w}{\sqrt{\Psi''(\lambda_*)(t-m)}}\right)dw \leq c_{15}\varepsilon^{1/12}\int_{-2\varepsilon}^{\infty} H(w+\varepsilon)\phi\left(\frac{w}{\sqrt{\Psi''(\lambda_*)t}}\right)dw + \frac{c_{14}(\varepsilon)}{t^q}\|H1_{[-\varepsilon,\infty)}\|_1.$$

Similarly, there exist constants $c_{16}$ (independent of $\varepsilon$) and $c_{17}(\varepsilon)$ such that

$$\int_{-\varepsilon}^{\infty} M_m(w)e^{-\frac{w^2}{2\Psi''(\lambda_*)t}}dw \leq c_{16}\varepsilon^{1/12}\int_{-2\varepsilon}^{\infty} H(w+\varepsilon)\phi\left(\frac{w}{\sqrt{\Psi''(\lambda_*)t}}\right)dw + \frac{c_{17}(\varepsilon)}{t^q}\|H1_{[-\varepsilon,\infty)}\|_1.$$

Combining the last two displays with (3.42) and (3.43), we get that there exist positive constants $c_{18}$ (independent of $\varepsilon$), $t_3(\varepsilon)$ and $c_{19}(\varepsilon)$ such that for $t > t_3(\varepsilon)$,

$$I_2^2(t) \leq \frac{c_{18}R^*(x)}{\sqrt{2\pi\Psi''(\lambda_*)t}}\varepsilon^{1/12}\int_{-2\varepsilon}^{\infty} H(w+\varepsilon)\phi\left(\frac{w}{\sqrt{\Psi''(\lambda_*)t}}\right)dw + c_{19}(\varepsilon)(1+x)\|H1_{[-\varepsilon,\infty)}\|_1\left(\frac{1}{t^{1+\varepsilon}}+\frac{1}{t^{1+\delta/2}}+\frac{1}{t^{1+q}}\right). \tag{3.46}$$

Combining (3.41) and (3.46), and using the fact that there exists $c_{20} > 0$ such that $R^*(x) \leq c_{20}(1+x)$, we get that there exist positive constants $c_{21}$ (independent of $\varepsilon$), $c_{22}(\varepsilon)$ and $t_4(\varepsilon)$, such that for $t > t_4(\varepsilon)$,

$$I_2(t) \leq c_{20}\varepsilon^{1/12}\frac{2R^*(x)}{\sqrt{2\pi\Psi''(\lambda_*)t}}\int_{-\varepsilon}^{\infty} H(u)\phi\left(\frac{u}{\sqrt{\Psi''(\lambda_*)t}}\right)du + c_{22}(\varepsilon)(1+x)\|H1_{[-\varepsilon,\infty)}\|_1\left(\frac{1}{t^{1+\varepsilon}}+\frac{1}{t^{1+\delta/2}}+\frac{1}{t^{1+q}}\right).$$

Combining this with (3.40) gives the desired result.

Now we prove the claims (3.44) and (3.45). Using the boundedness of $\rho$ and the fact that there exists a positive constant $t_5$ independent of $\varepsilon$ such that

$$\varepsilon^{1/6}\sqrt{[t]} - \varepsilon^{1/4}\sqrt{[t]} - 2\varepsilon > \frac{1}{2}\varepsilon^{1/6}\sqrt{[t]}, \quad t > t_5. \tag{3.47}$$

Therefore, we get that there exists a positive constant $c_{23}$ such that for $t > t_5$,

$$J_1(t) \leq c_{23} \int_{-2\varepsilon}^{\varepsilon^{1/4}\sqrt{[t]}} H(w+\varepsilon) \mathbf{P}_w^{\lambda_*}\left(\widehat{\xi}_m + 2\varepsilon > \varepsilon^{1/6}\sqrt{[t]}\right) dw \leq c_{23} \mathbf{P}_0^{\lambda_*}\left(\widehat{\xi}_m > \frac{1}{2}\varepsilon^{1/6}\sqrt{[t]}\right) \int_{-2\varepsilon}^{\varepsilon^{1/4}\sqrt{[t]}} H(w+\varepsilon) dw. \tag{3.48}$$

Using [11, Lemma 3.4] with $u = v = \frac{1}{2}\varepsilon^{1/6}\sqrt{[t]}$, since $m = [\varepsilon t]$, we get that there exist positive constants $c_{24}$ and $c_{25}$ both independent of $\varepsilon$ such that

$$\mathbf{P}_0^{\lambda_*}\left(\widehat{\xi}_m > \frac{1}{2}\varepsilon^{1/6}\sqrt{[t]}\right) \leq 2\exp\left\{\left(1 + \frac{4m}{\varepsilon^{1/3}[t]}\right)\right\} + m\mathbf{P}_0^{\lambda_*}\left(|\widehat{\xi}_1| > \frac{1}{2}\varepsilon^{1/6}\sqrt{[t]}\right) \leq c_{24}\varepsilon^{2/3} + [\varepsilon t]\frac{4\mathbf{E}_0^{\lambda_*}(\widehat{\xi}_1^2)}{\varepsilon^{1/3}[t]} \leq c_{25}\varepsilon^{1/6},$$

where in the second inequality we used Chebyshev's inequality and **(H1)**. Combining this with (3.48), we complete the proof of (3.44).

Next we prove (3.45). Using (3.47) and Hölder's inequality, we get that for all $t > t_5$ and $w > 0$, we have

$$\mathbf{P}_w^{\lambda_*}\left(\widehat{\xi}_m + 2\varepsilon > \varepsilon^{1/6}\sqrt{[t]}, \widehat{\tau}_0^- \leq m\right)$$

$$\leq \mathbf{P}_w^{\lambda_*}\left(\max_{s\in[0,m]}|\widehat{\xi}_s| > \frac{1}{2}\varepsilon^{1/6}\sqrt{[t]}\right)^{1/2} \mathbf{P}_w^{\lambda_*}\left(\widehat{\tau}_0^- \leq m\right)^{1/2}$$

$$= \mathbf{P}_w^{\lambda_*}\left(\max_{s\in[0,m]}|\widehat{\xi}_s| > \frac{1}{2}\varepsilon^{1/6}\sqrt{[t]}\right)^{1/2} \mathbf{P}_0^{\lambda_*}\left(\widehat{\tau}_{-w}^- \leq m\right)^{1/2}. \tag{3.49}$$

By Lemma 3.2, there exists a Brownian motion $W$ with diffusion coefficient $\Psi''(\lambda_*)$, starting from the origin, such that for any $t \geq 1$ and $x > 0$,

$$\mathbf{P}_0^{\lambda_*}(\widehat{A}_t) \leq \frac{C_3(2\varepsilon)}{t^{(\frac{1}{2}-2\varepsilon)(\delta+2)-1}},$$

where $\widehat{A}_t$ is defined by

$$\widehat{A}_t := \left\{\sup_{s\in[0,1]}|\widehat{\xi}_{ts} - \widehat{W}_{ts}| > t^{\frac{1}{2}-2\varepsilon}\right\}.$$

Therefore, there exists positive constant $q$ (independent of $\varepsilon$) and $c_{26}(\varepsilon)$ such that

$$\mathbf{P}_0^{\lambda_*}\left(\widehat{\tau}_{-w}^- \leq m, \widehat{A}_m\right) \leq \frac{C_3(2\varepsilon)}{m^{(\frac{1}{2}-2\varepsilon)(\delta+2)-1}} \leq \frac{c_{26}(\varepsilon)}{t^{2q}}. \tag{3.50}$$

Moreover, for $w > \varepsilon^{1/4}\sqrt{[t]}$, we have there exists a positive constant $c_{27}$ such that

$$\mathbf{P}_0^{\lambda_*}\left(\widehat{\tau}_{-w}^- \leq m, A_m^c\right) = \mathbf{P}_0^{\lambda_*}\left(\inf_{s\in[0,m]}\widehat{\xi}_s < -w, \widehat{A}_m^c\right)$$

$$\leq \mathbf{P}_0^{\lambda_*}\left(\inf_{s\in[0,m]}\widehat{W}_s < m^{\frac{1}{2}-2\varepsilon} - w\right) = \frac{2}{\sqrt{2\pi\Psi''(\lambda_*)m}} \int_{w-m^{\frac{1}{2}-2\varepsilon}}^{\infty} e^{-\frac{s^2}{2\Psi''(\lambda_*)m}} ds$$

$$\leq c_{27} \int_{\frac{w}{2\sqrt{\Psi''(\lambda_*)m}}}^{\infty} e^{-\frac{s^2}{2}} ds \leq \frac{2c_{27}\sqrt{\Psi''(\lambda_*)m}}{w} e^{-\frac{w^2}{8\Psi''(\lambda_*)m}}, \tag{3.51}$$

where in the last inequality we used the fact that $\int_a^\infty e^{-\frac{s^2}{2}} ds \leq \frac{1}{a}e^{-\frac{a^2}{2}}$ for any $a > 0$. Combining (3.51) and (3.50), for $w > \varepsilon^{1/4}\sqrt{[t]}$, since $\frac{\sqrt{m}}{w} \leq 1$, it holds that

$$\mathbf{P}_0^{\lambda_*}\left(\widehat{\tau}_{-w}^- \leq m\right)^{1/2} \leq c_{28}\phi\left(\frac{w}{\sqrt{\Psi''(\lambda_*)t}}\right) + \frac{c_{29}(\varepsilon)}{t^q},$$

for some positive constants $c_{28}$ and $c_{29}(\varepsilon)$. Similarly, we can get that

$$\mathbf{P}_w^{\lambda_*}\left(\max_{s\in[0,m]}|\widehat{\xi}_s| > \frac{1}{2}\varepsilon^{1/6}\sqrt{[t]}\right) \leq c_{30}\varepsilon^{1/6} + \frac{c_{31}(\varepsilon)}{t^{2q}},$$

for some positive constants $c_{30}$ and $c_{31}(\varepsilon)$. Combining this with (3.49), we get there exist positive constants $c_{32}$ and $c_{33}(\varepsilon)$ such that

$$\mathbf{P}_w^{\lambda_*}\left(\widehat{\xi}_m + 2\varepsilon > \varepsilon^{1/6}\sqrt{[t]}, \widehat{\tau}_0^- \leq m\right) \leq c_{32}\varepsilon^{1/12}\phi\left(\frac{w}{\sqrt{\Psi''(\lambda_*)t}}\right) + \frac{c_{33}(\varepsilon)}{t^q}, \quad w > \varepsilon^{1/4}\sqrt{[t]}.$$

This completes the proof of (3.45). $\quad\square$

**Lemma 3.8.** *Assume that $\xi$ is a Lévy process satisfying* **(H1)**, **(H2) (H3)** *and* $\mathbf{E}_0[\xi_1] < 0$. *Then one can find positive constants $C_{11}$ and $q$ with the property that for any $\varepsilon \in (0, \varepsilon_0)$ there exist positive constants $T_4(\varepsilon)$ and $C_{12}(\varepsilon)$ such that for any $x > 0$, $t > T_4(\varepsilon)$ and any Borel functions $h, H, g : \mathbb{R} \to \mathbb{R}_+$ satisfying $g \leq_\varepsilon h \leq_\varepsilon H$ and $\int_{\mathbb{R}_+} H(z - \varepsilon)(1 + z)dz < \infty$,*

$$
\mathbf{E}_x^{\lambda^*}\Big( h(\xi_t)1_{\{\tau_0^- > t\}} \Big) \geq \big( 1 - C_{11}t^{-1/2} - C_{12}(\varepsilon)t^{-\varepsilon} \big) \frac{2R^*(x)}{\sqrt{2\pi\Psi''(\lambda_*)^3}t^{3/2}} \int_{\mathbb{R}_+} g(z + \varepsilon)\widehat{R}^*(z)dz
$$
$$
- C_{12}(\varepsilon)\big( 1 + C_{11}\varepsilon t^{-1/2} + t^{-\varepsilon} \big) \frac{2R^*(x)}{\sqrt{2\pi\Psi''(\lambda_*)^3}t^{3/2}} \int_{\mathbb{R}_+} H(z - \varepsilon)\widehat{R}^*(z)dz
$$
$$
- \frac{C_{12}(\varepsilon)(1 + x)}{\sqrt{t}}\left( \frac{1}{t^{1+\varepsilon}} + \frac{1}{t^{1+\delta/2}} + \frac{1}{t^{1+q}} \right) \int_{\mathbb{R}_+} H(z - \varepsilon)(1 + z)dz.
$$

**Proof.** Recall that the functions $H_m$ and $I_m$ are defined in (3.29) and (3.30). Fix $\varepsilon \in (0, \varepsilon_0)$ and let $h, H, g : \mathbb{R} \to \mathbb{R}_+$ be Borel functions satisfying $g \leq_\varepsilon h \leq_\varepsilon H$ and $\int_{\mathbb{R}_+} H(z - \varepsilon)(1 + z)dz < \infty$. For any $y \in \mathbb{R}$, define

$$
N_m(y) := \mathbf{E}_y^{\lambda^*}\Big( g(\xi_m)1_{\{\xi_m \geq \varepsilon\}}1_{\{\tau_\varepsilon^- > m\}} \Big).
$$

Then for any $y > 0$ and $|v| \leq \varepsilon$,

$$
N_m(y) \leq \mathbf{E}_y^{\lambda^*}\Big( h(\xi_m + v)1_{\{\xi_m \geq \varepsilon\}}1_{\{\tau_\varepsilon^- > m\}} \Big) \leq \mathbf{E}_y^{\lambda^*}\Big( h(\xi_m + v)1_{\{\tau_{-v}^- > m\}} \Big) = I_m(y + v).
$$

Therefore, $N_m \leq_\varepsilon I_m \leq_\varepsilon H_m$. Applying Lemma 3.7 with $h = I_m$, we get that for $t - m > T_3(\varepsilon)$,

$$
\mathbf{E}_x^{\lambda_*}\Big( h(\xi_t)1_{\{\tau_0^- > t\}} \Big) = \mathbf{E}_x^{\lambda_*}\Big( I_m(\xi_{t-m}), \tau_0^- > t - m \Big)
$$
$$
\geq \frac{2R^*(x)}{\sqrt{2\pi}\Psi''(\lambda_*)(t - m)} \int_{\mathbb{R}_+} N_m(z)1_{\{z \geq \varepsilon\}}\rho\left( \frac{z}{\sqrt{\Psi''(\lambda_*)(t - m)}} \right)dz
$$
$$
- \frac{2C_9\varepsilon R^*(x)}{\sqrt{2\pi}\Psi''(\lambda_*)(t - m)} \int_{\mathbb{R}_+} I_m(z)\rho\left( \frac{z}{\sqrt{\Psi''(\lambda_*)(t - m)}} \right)dz
$$
$$
- \frac{C_9\varepsilon^{1/12}R^*(x)}{\sqrt{2\pi}\Psi''(\lambda_*)(t - m)} \int_{-\varepsilon}^\infty H_m(z)\phi\left( \frac{z}{\sqrt{\Psi''(\lambda_*)(t - m)}} \right)dz
$$
$$
- C_{10}(\varepsilon)(1 + x)\|H_m 1_{[-\varepsilon, \infty)}\|_1\left( \frac{1}{(t - m)^{1+\delta/2}} + \frac{1}{(t - m)^{1+\varepsilon}} + \frac{1}{(t - m)^{1+q}} \right) =: \sum_{i=1}^4 K_i,
$$

where $q$ is the constant in Lemma 3.7. By (3.15), we have

$$
K_1 = \frac{2R^*(x)}{\sqrt{2\pi}\Psi''(\lambda_*)(t - m)} \int_{\mathbb{R}_+} \mathbf{E}_z^{\lambda_*}\Big( g(\xi_m)1_{\{\xi_m \geq \varepsilon\}}1_{\{\tau_\varepsilon^- > m\}} \Big)1_{\{z \geq \varepsilon\}}\rho\left( \frac{z}{\sqrt{\Psi''(\lambda_*)(t - m)}} \right)dz
$$
$$
= \frac{2R^*(x)}{\sqrt{2\pi}\Psi''(\lambda_*)(t - m)} \int_{\mathbb{R}_+} g(z + \varepsilon)\mathbf{E}_z^{\lambda_*}\left( \rho\left( \frac{\widehat{\xi}_m + \varepsilon}{\sqrt{\Psi''(\lambda_*)(t - m)}} \right)1_{\{\widehat{\tau}_0^- > m\}} \right)dz.
$$

Repeating the argument leading to (3.36), we get that there exist positive constants $c_1$ (independent of $\varepsilon$) and $c_2(\varepsilon)$ such that

$$
K_1 \geq \big( 1 - c_1 t^{-1/2} \big) \frac{2R^*(x)}{\sqrt{2\pi\Psi''(\lambda_*)^3}t^{3/2}} \int_{\mathbb{R}_+} g(z + \varepsilon)\widehat{R}^*(z)dz - \frac{2c_2(\varepsilon)R^*(x)}{\sqrt{2\pi\Psi''(\lambda_*)^3}t^{3/2+\varepsilon}} \int_{\mathbb{R}_+} g(z + \varepsilon)(1 + z)dz. \tag{3.52}
$$

Using an argument similar to that leading to (3.36), we get that there exist positive constants $c_3$ independent of $\varepsilon$ and $c_4(\varepsilon)$ such that

$$
K_2 \geq -c_3\varepsilon(1 + c_3\varepsilon t^{-1/2}) \frac{2R^*(x)}{\sqrt{2\pi\Psi''(\lambda_*)^3}t^{3/2}} \int_{\mathbb{R}_+} H(z - \varepsilon)\widehat{R}^*(z)dz - c_4(\varepsilon) \frac{2R^*(x)}{\sqrt{2\pi\Psi''(\lambda_*)^3}t^{3/2+\varepsilon}} \int_{\mathbb{R}_+} H(z - \varepsilon)(1 + z)dz, \tag{3.53}
$$

and

$$
K_3 \geq -c_3\varepsilon^{1/12}(1 + c_3\varepsilon t^{-1/2}) \frac{2R^*(x)}{\sqrt{2\pi\Psi''(\lambda_*)^3}t^{3/2}} \int_{\mathbb{R}_+} H(z - \varepsilon)\widehat{R}^*(z)dz - c_4(\varepsilon) \frac{2R^*(x)}{\sqrt{2\pi\Psi''(\lambda_*)^3}t^{3/2+\varepsilon}} \int_{\mathbb{R}_+} H(z - \varepsilon)(1 + z)dz. \tag{3.54}
$$

Moreover, by (3.38), we get that

$$
K_4 \geq -\frac{c_5(\varepsilon)(1 + x)}{\sqrt{t}}\left( \frac{1}{t^{1+\varepsilon}} + \frac{1}{t^{1+\delta/2}} + \frac{1}{t^{1+q}} \right) \int_{\mathbb{R}_+} (1 + z)H(z - \varepsilon)dz, \tag{3.55}
$$

for some positive constant $c_5(\varepsilon)$. Combining (3.52), (3.53), (3.54) and (3.55), we get the desired result. $\square$

**Proof of Theorem 3.2:** Since $h : \mathbb{R} \to \mathbb{R}_+$ is a Borel function and $z \mapsto h(z)(1 + |z|)$ is directly Riemann integrable, by [11, Lemma 2.3], there exists $a \in (0, 1)$ such that $\int_{\mathbb{R}} \bar{h}_{a,\varepsilon}(1 + |z|)dz < \infty$, for any $\varepsilon \in (0, a)$, where $\bar{h}_{a,\varepsilon}$ is defined in (3.18). Applying Lemma 3.6 to $h$, we have for $t > T_2(\varepsilon)$,

$$t^{3/2}\mathbf{E}_x^{\lambda_*}\left(h(\xi_t)1_{\{\tau_0^->t\}}\right) \le \left(1 + C_7 t^{-1/2} + C_7\sqrt{\varepsilon}\right)\frac{2R^*(x)}{\sqrt{2\pi\Psi''(\lambda_*)^3}}\int_{\mathbb{R}_+}\bar{h}_{a_m,\varepsilon}(z-\varepsilon)\widehat{R}^*(z)\mathrm{d}z$$

$$+ \frac{2C_7 R^*(x)}{\sqrt{2\pi\Psi''(\lambda_*)^3}t^\varepsilon}\int_{\mathbb{R}_+}\bar{h}_{a_m,\varepsilon}(z-\varepsilon)(1+z)\mathrm{d}z$$

$$+ C_8(\varepsilon)(1+x)\left(\frac{1}{t^\varepsilon} + \frac{1}{t^{\delta/2}}\right)\int_{\mathbb{R}_+}\bar{h}_{a_m,\varepsilon}(z-\varepsilon)(1+z)\mathrm{d}z,$$

where $a_m = 2^{-m}a$, $m \ge 0$. On the other hand, by Lemma 3.8, we have for $t > T_4(\varepsilon)$,

$$t^{3/2}\mathbf{E}_x^{\lambda_*}\left(h(\xi_t)1_{\{\tau_0^->t\}}\right) \ge \left(1 - C_{11}t^{-1/2} - C_{12}(\varepsilon)t^{-\varepsilon}\right)\frac{2R^*(x)}{\sqrt{2\pi\Psi''(\lambda_*)^3}}\int_{\mathbb{R}_+}\underline{h}_{a_m,\varepsilon}(z+\varepsilon)\widehat{R}^*(z)\mathrm{d}z$$

$$- C_{11}\varepsilon\left(1 + C_{11}\varepsilon t^{-1/2} + t^{-\varepsilon}\right)\frac{2R^*(x)}{\sqrt{2\pi\Psi''(\lambda_*)^3}}\int_{\mathbb{R}_+}\bar{h}_{a_m,\varepsilon}(z-\varepsilon)\widehat{R}^*(z)\mathrm{d}z$$

$$- C_{12}(\varepsilon)(1+x)\left(\frac{1}{t^\varepsilon} + \frac{1}{t^{\delta/2}} + \frac{1}{t^q}\right)\int_{\mathbb{R}_+}(1+z)\bar{h}_{a_m,\varepsilon}(z-\varepsilon)\mathrm{d}z.$$

Since $h$ is not almost everywhere 0 on $(0,\infty)$, we have

$$\int_{\mathbb{R}_+}h(z)\widehat{R}^*(z)\mathrm{d}z \ge \widehat{R}^*(0)\int_{\mathbb{R}_+}h(z)\mathrm{d}z > 0.$$

Thus,

$$\limsup_{t\to\infty}\frac{t^{3/2}\mathbf{E}_x^{\lambda_*}\left(h(\xi_t)1_{\{\tau_0^->t\}}\right)}{\frac{2R^*(x)}{\sqrt{2\pi\Psi''(\lambda_*)^3}}\int_{\mathbb{R}_+}h(z)\widehat{R}^*(z)\mathrm{d}z} \le \left(1 + C_7\sqrt{\varepsilon}\right)\limsup_{t\to\infty}I(\varepsilon,m),\tag{3.56}$$

and

$$\limsup_{t\to\infty}\frac{t^{3/2}\mathbf{E}_x^{\lambda_*}\left(h(\xi_t)1_{\{\tau_0^->t\}}\right)}{\frac{2R^*(x)}{\sqrt{2\pi\Psi''(\lambda_*)^3}}\int_{\mathbb{R}_+}h(z)\widehat{R}^*(z)\mathrm{d}z} \ge \limsup_{t\to\infty}\left(J(\varepsilon,m) - C_{11}\varepsilon I(\varepsilon,m)\right),$$

where

$$I(\varepsilon,m) := \frac{\int_{\mathbb{R}_+}\bar{h}_{a_m,\varepsilon}(z-\varepsilon)\widehat{R}^*(z)\mathrm{d}z}{\int_{\mathbb{R}_+}h(z)\widehat{R}^*(z)\mathrm{d}z}, \quad J(\varepsilon,m) := \frac{\int_{\mathbb{R}_+}\underline{h}_{a_m,\varepsilon}(z+\varepsilon)(1+z)\mathrm{d}z}{\int_{\mathbb{R}_+}h(z)\widehat{R}^*(z)\mathrm{d}z}.$$

Repeating the argument for $I(y,\varepsilon,m)$ on [11, pp. 40–41], we get

$$\lim_{\varepsilon\to\infty}\limsup_{t\to\infty}I(\varepsilon,m) = 1.$$

This combined with (3.56) yields that

$$\limsup_{t\to\infty}t^{3/2}\mathbf{E}_x^{\lambda_*}\left(h(\xi_t)1_{\{\tau_0^->t\}}\right) \le \frac{2R^*(x)}{\sqrt{2\pi\Psi''(\lambda_*)^3}}\int_{\mathbb{R}_+}h(z)\widehat{R}^*(z)\mathrm{d}z.$$

The lower bound can be obtained in a similar way and this gives the desired result. □

## 4. Proof of Theorem 1.1 and Theorem 1.2

In this section, we give the proofs of Theorems 1.1 and 1.2. For any $x,t > 0$ and $y \ge 0$, define

$$u(x,t) := \mathbb{P}_x(\zeta > t),$$

and

$$Q_y(x,t) := \mathbb{P}_x(M_t > y).$$

It is easy to see that

$$Q_0(x,t) := \mathbb{P}_x(M_t > 0) = \mathbb{P}_x(\zeta > t) = u(x,t).$$

Let $B_b^+(\mathbb{R}_+)$ be the space of non-negative bounded Borel functions on $\mathbb{R}_+$. The following result is [1, Lemma 2.1] which is true for any branching killed Lévy process.

**Lemma 4.1.** *For any $h \in B_b^+(\mathbb{R}_+)$, the function*

$$u_h(x,t) := \mathbb{E}_x\left(e^{-\int_{\mathbb{R}_+} h(y)Z_t^0(dy)}\right), \quad t > 0, \ x > 0,$$

*solves the equation*

$$u_h(x,t) = \mathbf{E}_x\left(e^{-h(\xi_{t\wedge\tau_0^-})}\right) + \beta\mathbf{E}_x\left(\int_0^t \left(\sum_{k=0}^{\infty} p_k u_h(\xi_{s\wedge\tau_0^-}, t-s)^k - u_h(\xi_{s\wedge\tau_0^-}, t-s)\right)ds\right).$$

*Consequently, $v_h(x,t) = 1 - u_h(x,t)$ satisfies*

$$v_h(x,t) = \mathbf{E}_x\left(1 - e^{-h(\xi_{t\wedge\tau_0^-})}\right) - \mathbf{E}_x\left(\int_0^t \Phi(v_h(\xi_{s\wedge\tau_0^-}, t-s))ds\right).$$

The next result is also valid for any branching killed Lévy process.

**Lemma 4.2.** *For any $x,t > 0$ and $y \geq 0$, it holds that*

$$Q_y(x,t) = e^{-\alpha t}\mathbf{E}_x\left(1_{\{\tau_0^->t, \xi_t>y\}}e^{-\int_0^t \varphi(Q_y(\xi_s, t-s))ds}\right). \tag{4.1}$$

**Proof.** For any $x,t > 0$ and $y \geq 0$, by the dominated convergence we have

$$1 - Q_y(x,t) = \mathbb{P}_x(M_t \leq y) = \mathbb{P}_x(Z_t^0(y,\infty) = 0) = \lim_{\theta\to\infty}\mathbb{E}_x\left(e^{-\theta Z_t^0(y,\infty)}\right) = \lim_{\theta\to\infty}\mathbb{E}_x\left(e^{-\int_{\mathbb{R}_+}\theta 1_{(y,\infty)}(z)Z_t^0(dz)}\right).$$

Now applying Lemma 4.1 with $h(z) = 1_{(y,\infty)}(z)$, we get

$$Q_y(x,t) = \lim_{\theta\to\infty}\mathbf{E}_x\left(1 - e^{-\theta 1_{(y,\infty)}(\xi_{t\wedge\tau_0^-})}\right) - \mathbf{E}_x\left(\int_0^t \Phi(Q_y(\xi_{s\wedge\tau_0^-}, t-s))ds\right)$$

$$= \mathbf{P}_x\left(\xi_{t\wedge\tau_0^-} > y\right) - \mathbf{E}_x\left(\int_0^t \Phi(Q_y(\xi_{s\wedge\tau_0^-}, t-s))ds\right).$$

Thus $Q_y(x,t)$ is a bounded solution of the following equation

$$u(x,t) = \mathbf{P}_x\left(\xi_{t\wedge\tau_0^-} > y\right) - \mathbf{E}_x\left(\int_0^t \Phi(u(\xi_{s\wedge\tau_0^-}, t-s))ds\right). \tag{4.2}$$

It follows from [19, (4.8), p.102] that there is a unique positive locally bounded solution to (4.2). Thus we only need to prove that the right side of (4.1) is also a solution (4.2). For $s \in [0,t]$, define

$$A_{s,t} = -\int_s^t \frac{\Phi(Q_y(\xi_{r\wedge\tau_0^-}, t-r))}{Q_y(\xi_{r\wedge\tau_0^-}, t-r)}dr.$$

Note that $\frac{\Phi(u)}{u} = \varphi(u) + \alpha$ for $u \in (0,1]$. The right side of (4.1) can be written as $\mathbf{E}_x\left(e^{A_{0,t}}1_{\{\tau_0^->t, \xi_t>y\}}\right)$. It is elementary to check that

$$e^{A_{0,t}} = 1 - \int_0^t e^{A_{s,t}}\frac{\Phi(Q_y(\xi_{s\wedge\tau_0^-}, t-s))}{Q_y(\xi_{s\wedge\tau_0^-}, t-s)}ds.$$

Hence we have

$$\mathbf{E}_x\left(e^{A_{0,t}}1_{\{\tau_0^->t, \xi_t>y\}}\right) = \mathbf{P}_x\left(\xi_{t\wedge\tau_0^-} > y\right) - \mathbf{E}_x\left(1_{\{\tau_0^->t, \xi_t>y\}}\int_0^t e^{A_{s,t}}\frac{\Phi(Q_y(\xi_s, t-s))}{Q_y(\xi_s, t-s)}ds\right). \tag{4.3}$$

Now using the Markov property and the fact that

$$A_{s,t} = \int_0^{t-s} \frac{\Phi(Q_y(\xi_{(r+s)\wedge\tau_0^-}, t-r-s))}{Q_y(\xi_{(r+s)\wedge\tau_0^-}, t-r-s)}dr,$$

we see that (4.3) implies that $\mathbf{E}_x\left(e^{A_{0,t}}1_{\{\tau_0^->t, \xi_t>y\}}\right)$ solves (4.2). Thus, we have

$$Q_y(x,t) = \mathbf{E}_x\left(e^{A_{0,t}}1_{\{\tau_0^->t, \xi_t>y\}}\right) = e^{-\alpha t}\mathbf{E}_x\left(1_{\{\tau_0^->t, \xi_t>y\}}e^{-\int_0^t \varphi(Q_y(\xi_r, t-s))ds}\right).$$

This gives the desired result.  $\square$

The next lemma will be used to prove the lower bounds in Theorems 1.1 and 1.2.

**Lemma 4.3.** *Assume that (1.2) holds and $\xi$ is a Lévy process satisfying* **(H1)**. *Let $x > 0$.*

*(1) If $\mathbf{E}_0(\xi_1) = 0$, then for any $y \geq 0$, we have*

$$\liminf_{t\to\infty} \sqrt{t}e^{\alpha t}Q_{\sqrt{t}y}(x,t) \geq 2C_{sub}R(x)\phi_{\sigma^2}(y).$$

(2) If $\mathbf{E}_0(\xi_1) > 0$, we have

$$\liminf_{t\to\infty} e^{\alpha t} u(x,t) \geq q_x C_{sub}.$$

Moreover, for any $y \in \mathbb{R}$, we have

$$\liminf_{t\to\infty} e^{\alpha t} Q_{\sqrt{t}y+\mathbf{E}_0(\xi_1)t}(x,t) \geq q_x C_{sub} \int_{\frac{y}{\sigma}}^{\infty} \phi(z) \mathrm{d}z.$$

(3) If $\mathbf{E}_0(\xi_1) < 0$ and (H2) and (H3) hold, then for any $y \geq 0$, we have

$$\liminf_{t\to\infty} t^{3/2} e^{(\alpha-\Psi(\lambda_*))t} Q_y(x,t) \geq \frac{2C_{sub} R^*(x) e^{\lambda_* x}}{\sqrt{2\pi \Psi''(\lambda_*)^3}} \int_y^{\infty} e^{-\lambda_* z} \widehat{R}^*(z) \mathrm{d}z.$$

**Proof.** For any $y \geq 0$, by the definition of $Q$, we have

$$Q_y(x,t) \leq \mathbb{P}_x(\zeta > t) \leq \mathbb{P}_x(\widetilde{\zeta} > t) = g(t).$$

It follows from Lemma 4.2 that

$$Q_y(x,t) = e^{-\alpha t} \mathbf{E}_x \left( 1_{\{\tau_0^- > t, \xi_t > y\}} e^{-\int_0^t \varphi(Q_y(\xi_s, t-s)) \mathrm{d}s} \right)$$

$$\geq e^{-\alpha t} e^{-\int_0^t \varphi(g(t-s)) \mathrm{d}s} \mathbf{P}_x(\tau_0^- > t, \xi_t > y) \geq C_{sub} e^{-\alpha t} \mathbf{P}_x(\tau_0^- > t, \xi_t > y), \tag{4.4}$$

where the last inequality follows from (1.6). Applying Lemma 3.1, we immediately get the assertion of (1). Using the fact that $u(x,t) = Q_0(x,t)$, $\Psi'(0+) = \mathbb{E}_0(\xi_1)$ and applying (2.3) and (4.4), we get

$$\liminf_{t\to\infty} e^{\alpha t} u(x,t) \geq \liminf_{t\to\infty} C_{sub} \mathbf{P}_x(\tau_0^- > t) = q_x C_{sub}.$$

This gives the first result of (2). Applying Lemma 2.3 with $y$ replaced by $\sqrt{t}y + \mathbf{E}_0(\xi_1)t$, we get the second result of (2). Applying Theorem 3.1 with $f(x) = 1_{(y,\infty)}(x)$, we get the assertion of (3). $\quad\square$

In the following three lemmas, we prove the upper bounds in Theorems 1.1 and 1.2.

**Lemma 4.4.** Assume that (1.2) holds and $\xi$ is a Lévy process satisfying (H1). If $\mathbf{E}_0(\xi_1) = 0$, then for any $x > 0$ and $y \geq 0$, we have

$$\limsup_{t\to\infty} \sqrt{t} e^{\alpha t} Q_{\sqrt{t}y}(x,t) \leq 2C_{sub} R(x) \phi_{\sigma^2}(y).$$

**Proof.** Recall that $\varphi$ and $Q_y(\cdot, t)$ are increasing functions. Fix an $N > 0$. For $t > N$, by Lemma 4.2, we have

$$Q_{\sqrt{t}y}(x,t) \leq e^{-\alpha t} \mathbf{E}_x \left( 1_{\{\tau_0^- > t, \xi_t > \sqrt{t}y\}} e^{-\int_{t-N}^t \varphi(Q_{\sqrt{t}y}(\xi_s, t-s)) \mathrm{d}s} \right) \leq e^{-\alpha t} \mathbf{E}_x \left( 1_{\{\tau_0^- > t, \xi_t > \sqrt{t}y\}} e^{-\int_0^N \varphi(Q_{\sqrt{t}y}(\inf_{r\in[t-N,t]} \xi_r, s)) \mathrm{d}s} \right).$$

Take a $\gamma \in (0, \frac{1}{2})$ and define

$$J_1(t) := \mathbf{E}_x \left( 1_{\{\tau_0^- > t, \xi_t > \sqrt{t}y, \inf_{r\in[t-N,t]} \xi_r \geq \sqrt{t}y + t^\gamma\}} e^{-\int_0^N \varphi(Q_{\sqrt{t}y}(\inf_{r\in[t-N,t]} \xi_r, s)) \mathrm{d}s} \right),$$

$$J_2(t) := \mathbf{E}_x \left( 1_{\{\tau_0^- > t, \xi_t > \sqrt{t}y, \inf_{r\in[t-N,t]} \xi_r < \sqrt{t}y + t^\gamma\}} e^{-\int_0^N \varphi(Q_{\sqrt{t}y}(\inf_{r\in[t-N,t]} \xi_r, s)) \mathrm{d}s} \right).$$

Then $Q_{\sqrt{t}y}(x,t) \leq e^{-\alpha t}(J_1(t) + J_2(t))$. Since $Q_{\sqrt{t}y}(x,t)$ is increasing in $x$, we have

$$J_1(t) \leq e^{-\int_0^N \varphi(Q_{\sqrt{t}y}(\sqrt{t}y+t^\gamma, s)) \mathrm{d}s} \mathbf{P}_x(\tau_0^- > t, \xi_t > \sqrt{t}y). \tag{4.5}$$

By (4.4) and (1.5), we have

$$Q_{\sqrt{t}y}(x,t) \geq g(t) \mathbf{P}_x(\tau_0^- > t, \xi_t > \sqrt{t}y).$$

Thus,

$$e^{-\int_0^N \varphi(Q_{\sqrt{t}y}(\sqrt{t}y+t^\gamma, s)) \mathrm{d}s} \leq \exp\left\{ -\int_0^N \varphi\left( g(s) \mathbf{P}_{\sqrt{t}y+t^\gamma}(\tau_0^- > s, \xi_s > \sqrt{t}y) \right) \mathrm{d}s \right\}.$$

Plugging this into (4.5) and applying the dominated convergence theorem, we get

$$\limsup_{N\to\infty} \limsup_{t\to\infty} \frac{J_1(t)}{\mathbf{P}_x(\tau_0^- > t, \xi_t > \sqrt{t}y)} \leq \limsup_{N\to\infty} e^{-\int_0^N \varphi(g(s)) \mathrm{d}s} = C_{sub}. \tag{4.6}$$

Therefore, by Lemma 3.1, we have

$$\limsup_{N\to\infty} \limsup_{t\to\infty} \sqrt{t} J_1(t) \leq 2C_{sub} R(x) \phi_{\sigma^2}(y).$$

Now we show that

$$\lim_{t\to\infty}\sqrt{t}J_2(t)=0.\tag{4.7}$$

For any $\epsilon>0$ and $t>N$, it holds that

$$J_2(t)\le\mathbf{P}_x\left(\tau_0^->t,\xi_t>\sqrt{t}y,\inf_{r\in[t-N,t]}\xi_r<\sqrt{t}y+t^\gamma\right)$$

$$\le\mathbf{P}_x\left(\tau_0^->t,\sqrt{t}y<\xi_t\le\sqrt{t}(y+\epsilon)\right)+\mathbf{P}_x\left(\tau_0^->t,\xi_t>\sqrt{t}(y+\epsilon),\inf_{r\in[t-N,t]}\xi_r<\sqrt{t}y+t^\gamma\right).$$

By Lemma 3.1, we have

$$\lim_{t\to\infty}\sqrt{t}\mathbf{P}_x\left(\tau_0^->t,\sqrt{t}y<\xi_t\le\sqrt{t}(y+\epsilon)\right)=\frac{2R(x)}{\sqrt{2\pi\sigma^2}}\int_{\frac{y}{\sigma}}^{\frac{y+\epsilon}{\sigma}}\rho(z)\mathrm{d}z\xrightarrow{\epsilon\to0}0.$$

For any $t>0$ and $\kappa\in(0,\frac{\delta}{2(2+\delta)})$, define

$$A_t:=\left\{\sup_{s\in[0,1]}|\xi_{ts}-\xi_0-W_{ts}|>t^{\frac{1}{2}-\kappa}\right\},$$

where $W$ is the Brownian motion in Lemma 3.2. Then by the Markov property of $\xi$, for $k<t-N$,

$$\mathbf{P}_x\left(\tau_0^->t,\xi_t>\sqrt{t}(y+\epsilon),\inf_{r\in[t-N,t]}\xi_r<\sqrt{t}y+t^\gamma\right)$$

$$=\mathbf{E}_x\left(1_{\{\tau_0^->k\}}\mathbf{P}_{\xi_k}\left(\tau_0^->t-k,\xi_{t-k}>\sqrt{t}(y+\epsilon),\inf_{r\in[t-k-N,t-k]}\xi_r<\sqrt{t}y+t^\gamma\right)\right)\le H_1(t)+H_2(t),$$

where

$$H_1(t):=\mathbf{E}_x\left(1_{\{\tau_0^->k\}}\mathbf{P}_{\xi_k}\left(\tau_0^->t-k,\xi_{t-k}>\sqrt{t}(y+\epsilon),A_{t-k}\right)\right),$$

$$H_2(t):=\mathbf{E}_x\left(\mathbf{P}_{\xi_k}\left(\xi_{t-k}>\sqrt{t}(y+\epsilon),\inf_{r\in[t-k-N,t-k]}\xi_r<\sqrt{t}y+t^\gamma,A_{t-k}^c\right)\right).$$

To prove (4.7), we only need to prove

$$\limsup_{t\to\infty}\sqrt{t}H_1(t)=0,\quad\text{and}\quad\limsup_{t\to\infty}\sqrt{t}H_2(t)=0.\tag{4.8}$$

Using (3.23) and Lemma 3.2, we get that for any $k<t$,

$$H_1(t)\le\frac{CC_3(\kappa)}{(t-k)^{(\frac{1}{2}-\kappa)(\delta+2)-1}}\frac{1+x}{\sqrt{k}},$$

where $C>0$ is a constant. Taking $k=\frac{t}{2}$, we get that

$$\limsup_{t\to\infty}\sqrt{t}H_1(t)=0.\tag{4.9}$$

For any $z>0$, we have

$$\mathbf{P}_z\left(\xi_{t-k}>\sqrt{t}(y+\epsilon),\inf_{r\in[t-k-N,t-k]}\xi_r<\sqrt{t}y+t^\gamma,A_{t-k}^c\right)\le\mathbf{Q}_z\left(W_{t-k}>\sqrt{t}(y+\epsilon)-t^{\frac{1}{2}-\kappa},\inf_{r\in[t-k-N,t-k]}W_r<\sqrt{t}y+t^\gamma+t^{\frac{1}{2}-\kappa}\right),$$

where $(W_t,\mathbf{Q}_z)$ is a mean 0 Brownian motion with diffusion coefficient $\sigma^2$, starting from $z$. Therefore, for any $z>0$, using the reflection principle for Brownian motion, we get

$$\lim_{t\to\infty}\sqrt{t}\mathbf{P}_z\left(\xi_{t-k}>\sqrt{t}(y+\epsilon),\inf_{r\in[t-k-N,t-k]}\xi_r<\sqrt{t}y+t^\gamma,A_{t-k}^c\right)\le\lim_{t\to\infty}\sqrt{t}\mathbf{Q}_0\left(\inf_{r\in[0,N]}W_r<-\epsilon\sqrt{t}+t^\gamma+2t^{\frac{1}{2}-\kappa}\right)$$

$$=\lim_{t\to\infty}\sqrt{t}\mathbf{Q}_0\left(\max_{r\in[0,N]}W_r>\epsilon\sqrt{t}-t^\gamma-2t^{\frac{1}{2}-\kappa}\right)=0,$$

which implies that

$$\lim_{t\to\infty}\sqrt{t}H_2(t)=0.\tag{4.10}$$

Then (4.8) follows from (4.9) and (4.10), and we complete the proof. $\square$

**Lemma 4.5.** *Assume that* (1.2) *holds and that $\xi$ is a Lévy process satisfying* **(H1)**. *If $\mathbf{E}_0(\xi_1)>0$, then for any $x>0$, we have*

$$\limsup_{t\to\infty}e^{\alpha t}u(x,t)\le q_xC_{sub}.$$

*Moreover, for any $y\in\mathbb{R}$, we have*

$$\limsup_{t\to\infty}e^{\alpha t}Q_{\sqrt{t}y+\mathbf{E}_0(\xi_1)t}(x,t)\le q_xC_{sub}\int_{\frac{y}{\sigma}}^\infty\phi(z)\mathrm{d}z.$$

**Proof.** Take $\gamma \in (0, \frac{1}{2})$ and fix an $N > 0$. For $t > N$, using Lemma 4.2, we have

$$Q_{\sqrt{t}y + \mathbf{E}_0(\xi_1)t}(x, t) \leq e^{-\alpha t} \mathbf{E}_x \left( 1_{\{\tau_0^- > t, \xi_t > \sqrt{t}y + \mathbf{E}_0(\xi_1)t\}} e^{-\int_{t-N}^t \varphi(Q_{\sqrt{t}y + \mathbf{E}_0(\xi_1)t}(\xi_s, t-s)) ds} \right)$$

$$\leq e^{-\alpha t} \mathbf{E}_x \left( 1_{\{\tau_0^- > t, \xi_t > \sqrt{t}y + \mathbf{E}_0(\xi_1)t\}} e^{-\int_0^N \varphi(Q_{\sqrt{t}y + \mathbf{E}_0(\xi_1)t}(\inf_{r \in [t-N,t]} \xi_r, s)) ds} \right)$$

$$=: e^{-\alpha t}(K_1(t) + K_2(t)),$$

where

$$K_1(t) := \mathbf{E}_x \left( 1_{\{\tau_0^- > t, \xi_t > \sqrt{t}y + \mathbf{E}_0(\xi_1)t, \inf_{r \in [t-N,t]} \xi_r \geq \sqrt{t}y + \mathbf{E}_0(\xi_1)t + t^\gamma\}} e^{-\int_0^N \varphi(Q_{\sqrt{t}y + \mathbf{E}_0(\xi_1)t}(\inf_{r \in [t-N,t]} \xi_r, s)) ds} \right),$$

$$K_2(t) := \mathbf{E}_x \left( 1_{\{\tau_0^- > t, \xi_t > \sqrt{t}y + \mathbf{E}_0(\xi_1)t, \inf_{r \in [t-N,t]} \xi_r < \sqrt{t}y + \mathbf{E}_0(\xi_1)t + t^\gamma\}} e^{-\int_0^N \varphi(Q_{\sqrt{t}y + \mathbf{E}_0(\xi_1)t}(\inf_{r \in [t-N,t]} \xi_r, s)) ds} \right).$$

Repeating the argument leading to (4.6), we obtain that

$$\limsup_{N \to \infty} \limsup_{t \to \infty} \frac{K_1(t)}{\mathbf{P}_x\left(\tau_0^- > t, \xi_t > \sqrt{t}y + \mathbf{E}_0(\xi_1)t\right)} \leq C_{sub}. \tag{4.11}$$

Therefore, by Lemma 2.3, we have

$$\limsup_{N \to \infty} \limsup_{t \to \infty} K_1(t) \leq q_x C_{sub} \int_{\frac{y}{\sigma}}^\infty \phi(z) dz.$$

Next, we show that $\lim_{t \to \infty} K_2(t) = 0$. For $\epsilon > 0$, it holds that

$$K_2(t) \leq \mathbf{P}_x \left( \tau_0^- > t, \xi_t > \sqrt{t}y + \mathbf{E}_0(\xi_1)t, \inf_{r \in [t-N,t]} \xi_r < \sqrt{t}y + \mathbf{E}_0(\xi_1)t + t^\gamma \right)$$

$$\leq \mathbf{P}_x \left( \tau_0^- > t, \sqrt{t}y + \mathbf{E}_0(\xi_1)t < \xi_t \leq \sqrt{t}(y+\epsilon) + \mathbf{E}_0(\xi_1)t \right) + \mathbf{P}_x \left( \xi_t > \sqrt{t}(y+\epsilon) + \mathbf{E}_0(\xi_1)t, \inf_{r \in [t-N,t]} \xi_r < \sqrt{t}y + \mathbf{E}_0(\xi_1)t + t^\gamma \right).$$

By Lemma 2.3, we have

$$\lim_{t \to \infty} \mathbf{P}_x \left( \tau_0^- > t, \sqrt{t}y + \mathbf{E}_0(\xi_1)t < \xi_t \leq \sqrt{t}(y+\epsilon) + \mathbf{E}_0(\xi_1)t \right) \xrightarrow{\epsilon \to 0} 0. \tag{4.12}$$

For $\mathbf{E}_0(\xi_1) > 0$, since $((\xi_t - \mathbf{E}_0(\xi_1)t)_{t \geq 0}, (\mathbf{P}_x)_{x \in \mathbb{R}})$ is a Lévy process satisfying $\mathbf{E}_0(\xi_1 - \mathbf{E}_0(\xi_1)) = 0$, it follows from Lemma 3.2 that there exists a Brownian motion $W$ with diffusion coefficient $\sigma^2 = \mathbf{E}_0(\xi_1^2)$ starting from the origin such that for all $t \geq 1$,

$$\mathbf{P}_x \left( \sup_{s \in [0,1]} |(\xi_{ts} - \mathbf{E}_0(\xi_1)ts) - x - W_{ts}| > t^{\frac{1}{2} - \kappa} \right) \leq \frac{C_3(\kappa)}{t^{(\frac{1}{2} - \kappa)(\delta+2) - 1}}, \tag{4.13}$$

where $\kappa$ and $C_3(\kappa)$ are defined in Lemma 3.2. Let

$$D_t := \left\{ \sup_{s \in [0,1]} |(\xi_{ts} - \mathbf{E}_0(\xi_1)ts) - x - W_{ts}| > t^{\frac{1}{2} - \kappa} \right\},$$

then by (4.13), we have $\lim_{t \to \infty} \mathbf{P}_x(D_t) = 0$. Moreover, we have

$$\mathbf{P}_x \left( \xi_t > \sqrt{t}(y+\epsilon) + \mathbf{E}_0(\xi_1)t, \inf_{r \in [t-N,t]} \xi_r < \sqrt{t}y + \mathbf{E}_0(\xi_1)t + t^\gamma \right)$$

$$\leq \mathbf{P}_x(D_t) + \mathbf{P}_x \left( \xi_t > \sqrt{t}(y+\epsilon) + \mathbf{E}_0(\xi_1)t, \inf_{r \in [t-N,t]} \xi_r < \sqrt{t}y + \mathbf{E}_0(\xi_1)t + t^\gamma, D_t^c \right).$$

Furthermore, using the reflection principle for Brownian motion, we get that, as $t \to \infty$,

$$\mathbf{P}_x \left( \xi_t > \sqrt{t}(y+\epsilon) + \mathbf{E}_0(\xi_1)t, \inf_{r \in [t-N,t]} \xi_r < \sqrt{t}y + \mathbf{E}_0(\xi_1)t + t^\gamma, D_t^c \right)$$

$$\leq \mathbf{Q}_x \left( W_t > \sqrt{t}(y+\epsilon) - t^{\frac{1}{2} - \kappa}, \inf_{r \in [t-N,t]} W_r < \sqrt{t}y + t^\gamma + t^{\frac{1}{2} - \kappa} \right)$$

$$\leq \mathbf{Q}_0 \left( \inf_{r \in [0,N]} W_r < -\epsilon\sqrt{t} + t^\gamma + 2t^{\frac{1}{2} - \kappa} \right) = \mathbf{Q}_0 \left( \max_{r \in [0,N]} W_r > \epsilon\sqrt{t} - t^\gamma - 2t^{\frac{1}{2} - \kappa} \right) \to 0,$$

where $(W_t, \mathbf{Q}_x)$ is a mean 0 Brownian motion with diffusion coefficient $\sigma^2$, starting from $x$. This combined with (4.11) and (4.12) gives the desired result. $\square$

**Lemma 4.6.** *Fix an $N > 0$. Assume that (1.2) holds and $\xi$ is a Lévy process satisfying* **(H1)**, **(H2)** *and* **(H3)**. *If $\mathbf{E}_0(\xi_1) < 0$, then we have*

$$\lim_{t \to \infty} t^{3/2} e^{-\Psi(\lambda_*)t} \mathbf{E}_x \left( 1_{\{\tau_0^- > t, \xi_t > y\}} e^{-\int_{t-N}^t \varphi(Q_y(\xi_s, t-s)) ds} \right) = e^{(\alpha - \Psi(\lambda_*))N} \frac{2R^*(x)e^{\lambda_* x}}{\sqrt{2\pi\Psi''(\lambda_*)^3}} \int_{\mathbb{R}_+} \mathbb{P}_z(M_N > y) e^{-\lambda_* z} \widehat{R}^*(z) dz.$$

**Proof.** By the Markov property,

$$\mathbf{E}_x\left(1_{\{\tau_0^->t,\xi_t>y\}}e^{-\int_{t-N}^t \varphi(Q_y(\xi_s,t-s))ds}\right) = \mathbf{E}_x\left(1_{\{\tau_0^->t-N\}}\mathbf{E}_{\xi_{t-N}}\left(1_{\{\tau_0^->N,\xi_N>y\}}e^{-\int_0^N \varphi(Q_y(\xi_s,N-s))ds}\right)\right) =: \mathbf{E}_x\left(1_{\{\tau_0^->t-N\}}f_N^y(\xi_{t-N})\right),$$

where for any $z \geq 0$, $f_N^y$ is defined by

$$f_N^y(z) := \mathbf{E}_z\left(1_{\{\tau_0^->N,\xi_N>y\}}e^{-\int_0^N \varphi(Q_y(\xi_s,N-s))ds}\right).$$

By Lemma 4.2,

$$f_N^y(z) = e^{\alpha N}Q_y(z,N), \tag{4.14}$$

which implies that $f_N^y(z)$ is bounded and increasing with respect to $z$. Then $f_N^y$ is a.e. continuous. By [20, Corollary 3.2], $f_N^y(z)e^{-\lambda_*z}(1+|z|)$ is directly Riemann integrable with respect to $z$. Applying Theorem 3.1 with $f$ replaced by $f_N^y$, we get

$$\lim_{t\to\infty} t^{3/2}e^{-\Psi(\lambda_*)t}\mathbf{E}_x\left(1_{\{\tau_0^->t,\xi_t>y\}}e^{-\int_{t-N}^t \varphi(Q_y(\xi_s,t-s))ds}\right) = e^{-\Psi(\lambda_*)N}\frac{2R^*(x)e^{\lambda_*x}}{\sqrt{2\pi\Psi''(\lambda_*)^3}}\int_{\mathbb{R}_+} f_N^y(z)e^{-\lambda_*z}\widehat{R}^*(z)dz,$$

which gives the desired result together with (4.14). $\square$

**Proofs of Theorems 1.1 and 1.2:** Combining Lemmas 4.3, 4.4 and 4.5, we get parts (1) and (2) of both Theorem 1.1 and Theorem 1.2 immediately. Next, we prove part (3) of both theorems. For $\mathbf{E}_0[\xi_1] < 0$, fix $N > 0$ and $y \geq 0$. By Lemma 4.2, we have for $t \geq N$,

$$Q_y(x,t) \leq e^{-\alpha t}\mathbf{E}_x\left(1_{\{\tau_0^->t,\xi_t>y\}}e^{-\int_{t-N}^t \varphi(Q_y(\xi_s,t-s))ds}\right). \tag{4.15}$$

Combining this with Lemma 4.6, we get that

$$\limsup_{t\to\infty} t^{3/2}e^{(\alpha-\Psi(\lambda_*))t}Q_y(x,t) \leq \frac{2R^*(x)e^{\lambda_*x}}{\sqrt{2\pi\Psi''(\lambda_*)^3}}\liminf_{N\to\infty} e^{(\alpha-\Psi(\lambda_*))N}\int_{\mathbb{R}_+} \mathbb{P}_z(M_N>y)e^{-\lambda_*z}\widehat{R}^*(z)dz. \tag{4.16}$$

Moreover, using the fact that $Q_y(x,t) \leq g(t) = \mathbb{P}_0(\widetilde{\zeta} > t)$ and Lemma 4.2, we get

$$Q_y(x,t) \geq e^{-\alpha t}\mathbf{E}_x\left(1_{\{\tau_0^->t,\xi_t>y\}}e^{-\int_{t-N}^t \varphi(Q_y(\xi_s,t-s))ds}\right)e^{-\int_0^{t-N} \varphi(g(t-s))ds}.$$

Using Lemma 4.6 again, we have

$$\liminf_{t\to\infty} t^{3/2}e^{(\alpha-\Psi(\lambda_*))t}Q_y(x,t) \geq \frac{2R^*(x)e^{\lambda_*x}}{\sqrt{2\pi\Psi''(\lambda_*)^3}}\limsup_{N\to\infty} e^{(\alpha-\Psi(\lambda_*))N}\int_{\mathbb{R}_+} \mathbb{P}_z(M_N>y)e^{-\lambda_*z}\widehat{R}^*(z)dz. \tag{4.17}$$

Combining (4.16) and (4.17), we obtain that

$$\lim_{t\to\infty} t^{3/2}e^{(\alpha-\Psi(\lambda_*))t}Q_y(x,t) = \frac{2R^*(x)e^{\lambda_*x}}{\sqrt{2\pi\Psi''(\lambda_*)^3}}\lim_{N\to\infty} e^{(\alpha-\Psi(\lambda_*))N}\int_{\mathbb{R}_+} \mathbb{P}_z(M_N>y)e^{-\lambda_*z}\widehat{R}^*(z)dz := \frac{2R^*(x)e^{\lambda_*x}}{\sqrt{2\pi\Psi''(\lambda_*)^3}}C_y,$$

where $C_y := \lim_{N\to\infty} e^{(\alpha-\Psi(\lambda_*))N}\int_{\mathbb{R}_+} \mathbb{P}_z(M_N>y)e^{-\lambda_*z}\widehat{R}^*(z)dz$. Next, we show that $C_y \in (0,\infty)$. First, applying Lemma 4.3 (3), we get

$$C_y \geq C_{sub}\int_y^\infty e^{-\lambda_*z}\widehat{R}^*(z)dz > 0.$$

Using (4.15) and taking $f(x) = 1_{(y,\infty)}(x)$ in Theorem 3.1, we get

$$\limsup_{t\to\infty} t^{3/2}e^{(\alpha-\Psi(\lambda_*))t}Q_y(x,t) \leq \lim_{t\to\infty} t^{3/2}e^{-\Psi(\lambda_*)t}\mathbf{P}_x\left(\tau_0^->t,\xi_t>y\right) = \frac{2R^*(x)e^{\lambda_*x}}{\sqrt{2\pi\Psi''(\lambda_*)^3}}\int_y^\infty e^{-\lambda_*z}\widehat{R}^*(z)dz.$$

Therefore, $C_y \leq \int_y^\infty e^{-\lambda_*z}\widehat{R}^*(z)dz < \infty$. This completes the proof. $\square$

**Proof of Corollary 1.1:** We only prove (3). Combining Theorems 1.1 and 1.2, for any $0 < a < b$, we get that

$$\lim_{t\to\infty} \mathbb{P}_x\left(M_t \in (a,b]|\zeta>t\right) = \lim_{t\to\infty} \frac{Q_a(x,t)-Q_b(x,t)}{u(x,t)} = \frac{\lim_{N\to\infty}\int_0^\infty \mathbb{P}_z(M_N\in(a,b])e^{-\lambda_*z}\widehat{R}^*(z)dz}{\lim_{N\to\infty}\int_0^\infty \mathbb{P}_z(M_N\in(0,\infty))e^{-\lambda_*z}\widehat{R}^*(z)dz}.$$

Therefore, there exists a random variable $(X,\mathbb{P})$ such that $\mathbb{P}_x(M_t \in \cdot|\zeta>t)$ vaguely converge to $\mathbb{P}(X\in\cdot)$. Moreover, by (4.4), we have

$$\mathbb{P}_x\left(M_t>y|\zeta>t\right) = \frac{Q_y(x,t)}{u(x,t)} \leq \frac{e^{-\alpha t}\mathbf{P}_x\left(\tau_0^->t,\xi_t>y\right)}{C_{sub}e^{-\alpha t}\mathbf{P}_x\left(\tau_0^->t\right)} = \frac{1}{C_{sub}}\mathbf{P}_x\left(\xi_t>y|\tau_0^->t\right).$$

Thus by Theorem 3.1, the tightness of $M_t$ under $\mathbb{P}_x(\cdot|\zeta>t)$ follows from the tightness of $\xi_t$ under $\mathbf{P}_x(\cdot|\tau_0^->t)$. This gives the desired result. $\square$

## 5. Proof of Theorem 1.3

Recall that $\alpha = \beta(1 - m)$. For any $0 < x < y$, define

$$v(x, y) := \mathbb{P}_x(M > y).$$

The following result is valid for any branching killed Lévy processes.

**Lemma 5.1.** *For any $0 < x < y$, it holds that*

$$v(x, y) = \mathbf{E}_x\left(1_{\{\tau_y^+ < \tau_0^-\}} e^{-\alpha\tau_y^+ - \int_0^{\tau_y^+} \varphi(v(\xi_s, y))ds}\right),$$

*where $\varphi$ is defined by (1.1). Consequently, for $0 < x < z < y$, by the strong Markov property, we have*

$$v(x, y) = \mathbf{E}_x\left(1_{\{\tau_z^+ < \tau_0^-\}} v(\xi_{\tau_z^+}, y) e^{-\alpha\tau_z^+ - \int_0^{\tau_z^+} \varphi(v(\xi_s, y))ds}\right).$$

**Proof.** For $0 < x < y$, comparing the first branching time with $\tau_y^+$, we have

$$v(x, y) = \int_0^\infty \beta e^{-\beta s} \mathbf{P}_x(\tau_y^+ < \tau_0^-, \tau_y^+ \leq s)ds + \int_0^\infty \beta e^{-\beta s} \mathbf{E}_x\left(\left(1 - \sum_{k=0}^\infty p_k(1 - v(\xi_s, y))^k\right)1_{\{\tau_y^+ \wedge \tau_0^- > s\}}\right)ds$$

$$= \mathbf{E}_x\left(e^{-\beta\tau_y^+} 1_{\{\tau_y^+ < \tau_0^-\}}\right) + \int_0^\infty \beta e^{-\beta s} \mathbf{E}_x\left(\left(1 - \sum_{k=0}^\infty p_k(1 - v(\xi_s, y))^k\right)1_{\{\tau_y^+ \wedge \tau_0^- > s\}}\right)ds.$$

By [21, Lemma 4.1], the above equation is equivalent to

$$v(x, y) + \beta\int_0^\infty \mathbf{E}_x\left(v(\xi_s, y)1_{\{\tau_y^+ \wedge \tau_0^- > s\}}\right)ds = \mathbf{P}_x\left(\tau_y^+ < \tau_0^-\right) + \beta\int_0^\infty \mathbf{E}_x\left(\left(1 - \sum_{k=0}^\infty p_k(1 - v(\xi_s, y))^k\right)1_{\{\tau_y^+ \wedge \tau_0^- > s\}}\right)ds,$$

which is also equivalent to

$$v(x, y) = \mathbf{P}_x\left(\tau_y^+ < \tau_0^-\right) - \mathbf{E}_x\left(\int_0^{\tau_y^+ \wedge \tau_0^-} \Phi(v(\xi_s, y))ds\right),$$

where $\Phi$ is defined in (1.1). By repeating the argument leading to (4.1), we get the desired result. $\square$

In the remainder of this section, we always assume that $((\xi_t)_{t\geq 0}, (\mathbf{P}_x)_{x\in\mathbb{R}})$ is a spectrally negative Lévy process.

**Lemma 5.2.** *Assume that $\xi$ is a spectrally negative Lévy process. For any $0 < x < y$, we have*

$$v(x, y) \leq \frac{e^{x\psi(\alpha)}W_{\psi(\alpha)}^{(0)}(x)}{e^{y\psi(\alpha)}W_{\psi(\alpha)}^{(0)}(y)} \leq e^{(x-y)\psi(\alpha)}. \tag{5.1}$$

**Proof.** Since the function $\varphi$ is non-negative, combining Lemma 5.1 and Theorem 2.1 (2), we get

$$v(x, y) \leq \mathbf{E}_x\left(e^{-\alpha\tau_y^+} 1_{\{\tau_y^+ < \tau_0^-\}}\right) = \frac{W^{(\alpha)}(x)}{W^{(\alpha)}(y)}, \quad x < y.$$

This combined with Lemma 2.1 yields that

$$v(x, y) \leq \frac{e^{x\psi(\alpha)}W_{\psi(\alpha)}^{(0)}(x)}{e^{y\psi(\alpha)}W_{\psi(\alpha)}^{(0)}(y)} \leq e^{(x-y)\psi(\alpha)}.$$

This gives the desired result. $\square$

**Proof of Theorem 1.3:** By Lemmas 5.1 and 2.2, we have

$$v(x, y) = \mathbf{E}_x\left(1_{\{\tau_y^+ < \tau_0^-\}} e^{-\alpha\tau_y^+ - \int_0^{\tau_y^+} \varphi(v(\xi_s, y))ds}\right) = e^{(x-y)\psi(\alpha)}\mathbf{E}_x^{\psi(\alpha)}\left(1_{\{\tau_y^+ < \tau_0^-\}} e^{-\int_0^{\tau_y^+} \varphi(v(\xi_s, y))ds}\right).$$

Fix a $\gamma \in (0, 1)$, by the Markov property of $(\xi_t, \mathbf{P}_x^{\psi(\alpha)})$, we have

$$v(x, y) = e^{(x-y)\psi(\alpha)}\mathbf{E}_x^{\psi(\alpha)}\left(1_{\{\tau_{y-y^\gamma}^+ < \tau_0^-\}} e^{-\int_0^{\tau_{y-y^\gamma}^+} \varphi(v(\xi_s, y))ds}\right)\mathbf{E}_{y-y^\gamma}^{\psi(\alpha)}\left(1_{\{\tau_y^+ < \tau_0^-\}} e^{-\int_0^{\tau_y^+} \varphi(v(\xi_s, y))ds}\right) =: e^{(x-y)\psi(\alpha)} A_1(x, y) A_2(y), \tag{5.2}$$

where

$$A_1(x, y) := \mathbf{E}_x^{\psi(\alpha)}\left(1_{\{\tau_{y-y^\gamma}^+ < \tau_0^-\}} e^{-\int_0^{\tau_{y-y^\gamma}^+} \varphi(v(\xi_s, y))ds}\right),$$

$$A_2(y) := \mathbf{E}_{y-y^\gamma}^{\psi(\alpha)}\left(1_{\{\tau_y^+ < \tau_0^-\}}e^{-\int_0^{\tau_y^+}\varphi(v(\xi_s,y))\mathrm{d}s}\right).$$

We first consider the asymptotic behavior of $A_1(x, y)$ as $y \to \infty$. We claim that

$$\lim_{y\to\infty}\left(\mathbf{P}_x^{\psi(\alpha)}(\tau_{y-y^\gamma}^+ < \tau_0^-) - A_1(x, y)\right) = 0. \tag{5.3}$$

Indeed, using the inequality $1 - e^{-|x|} \le |x|$, we get

$$0 \le \mathbf{P}_x^{\psi(\alpha)}(\tau_{y-y^\gamma}^+ < \tau_0^-) - A_1(x, y) = \mathbf{E}_x^{\psi(\alpha)}\left(1_{\{\tau_{y-y^\gamma}^+ < \tau_0^-\}}\left(1 - e^{-\int_0^{\tau_{y-y^\gamma}^+}\varphi(v(\xi_s,y))\mathrm{d}s}\right)\right)$$

$$\le \mathbf{E}_x^{\psi(\alpha)}\left(1_{\{\tau_{y-y^\gamma}^+ < \tau_0^-\}}\int_0^{\tau_{y-y^\gamma}^+}\varphi(v(\xi_s,y))\mathrm{d}s\right) \le \mathbf{E}_x^{\psi(\alpha)}\left(\int_0^{\tau_{y-y^\gamma}^+}\varphi(v(\xi_s,y))\mathrm{d}s\right). \tag{5.4}$$

Set $y_*(x) := \inf\{t \ge y - y^\gamma, t - x \in \mathbb{N}\}$. By (5.1), we have

$$\mathbf{E}_x^{\psi(\alpha)}\left(\int_0^{\tau_{y-y^\gamma}^+}\varphi(v(\xi_s,y))\mathrm{d}s\right) \le \mathbf{E}_x^{\psi(\alpha)}\left(\int_0^{\tau_{y_*(x)}^+}\varphi\left(e^{(\xi_s-y)\psi(\alpha)}\right)\mathrm{d}s\right) = \sum_{k=0}^{y_*(x)-x-1}\mathbf{E}_x^{\psi(\alpha)}\left(\int_{\tau_{x+k}^+}^{\tau_{x+k+1}^+}\varphi\left(e^{(\xi_s-y)\psi(\alpha)}\right)\mathrm{d}s\right)$$

$$\le \sum_{k=0}^{y_*(x)-x-1}\mathbf{E}_x^{\psi(\alpha)}\left(\tau_{x+k+1}^+ - \tau_{x+k}^+\right)\varphi\left(e^{\psi(\alpha)(x+k+1-y)}\right) = \mathbf{E}_0^{\psi(\alpha)}\left(\tau_1^+\right)\sum_{k=1}^{y_*(x)}\varphi\left(e^{-\psi(\alpha)(y-x-1-y_*(x)+k)}\right).$$

By the definition of $y_*(x)$, we have that for $y$ large enough,

$$y - x - 1 - y_*(x) \ge y - x - 1 - (y - y^\gamma + 1) = y^\gamma - x - 2.$$

Therefore, when $y$ is sufficient large so that $y^\gamma - x - 2 \ge y^{\gamma/2}$, by (1.3), we have

$$\mathbf{E}_x^{\psi(\alpha)}\left(\int_0^{\tau_{y-y^\gamma}^+}\varphi(v(\xi_s,y))\mathrm{d}s\right) \le \mathbf{E}_0^{\psi(\alpha)}\left(\tau_1\right)\sum_{k=1}^\infty\varphi\left(e^{-\psi(\alpha)(y^{\gamma/2}+k)}\right)$$

$$\le \mathbf{E}_0^{\psi(\alpha)}\left(\tau_1\right)\int_0^\infty\varphi\left(e^{-\psi(\alpha)(y^{\gamma/2}+z)}\right)\mathrm{d}z = \mathbf{E}_0^{\psi(\alpha)}\left(\tau_1\right)\int_{y^{\gamma/2}}^\infty\varphi\left(e^{-\psi(\alpha)z}\right)\mathrm{d}z \xrightarrow{y\to\infty} 0.$$

This combined with (5.4) yields (5.3). Using Lemma 2.2 and Theorem 2.1(2), we get

$$\lim_{y\to\infty}A_1(x, y) = \lim_{y\to\infty}\mathbf{P}_x^{\psi(\alpha)}(\tau_{y-y^\gamma}^+ < \tau_0^-)$$

$$= \lim_{y\to\infty}e^{(y-y^\gamma-x)\psi(\alpha)}\mathbf{E}_x\left(e^{-\alpha\tau_{y-y^\gamma}^+}1_{\{\tau_{y-y^\gamma}^+ < \tau_0^-\}}\right) = \lim_{y\to\infty}e^{(y-y^\gamma-x)\psi(\alpha)}\frac{W^{(\alpha)}(x)}{W^{(\alpha)}(y-y^\gamma)}.$$

Using (2.1), we get that

$$W^{(\alpha)}(y - y^\gamma) \sim \frac{e^{\psi(\alpha)(y-y^\gamma)}}{\Psi'(\psi(\alpha))}, \quad \text{as } y \to \infty.$$

Therefore,

$$\lim_{y\to\infty}A_1(x, y) = e^{-\psi(\alpha)x}\Psi'(\psi(\alpha))W^{(\alpha)}(x). \tag{5.5}$$

Next, we consider the asymptotic behavior of $A_2(y)$ as $y \to \infty$. Recall that

$$A_2(y) = \mathbf{E}_{y-y^\gamma}^{\psi(\alpha)}\left(1_{\{\tau_y^+ < \tau_0^-\}}e^{-\int_0^{\tau_y^+}\varphi(v(\xi_s,y))\mathrm{d}s}\right)$$

$$= \mathbf{E}_{y-y^\gamma}^{\psi(\alpha)}\left(e^{-\int_0^{\tau_y^+}\varphi(v(\xi_s,y))\mathrm{d}s}\right) - \mathbf{E}_{y-y^\gamma}^{\psi(\alpha)}\left(1_{\{\tau_y^+ \ge \tau_0^-\}}e^{-\int_0^{\tau_y^+}\varphi(v(\xi_s,y))\mathrm{d}s}\right).$$

We claim that

$$\lim_{y\to\infty}\mathbf{E}_{y-y^\gamma}^{\psi(\alpha)}\left(e^{-\int_0^{\tau_y^+}\varphi(v(\xi_s,y))\mathrm{d}s}\right) = C_*(\alpha) \in (0, 1], \tag{5.6}$$

and

$$\lim_{y\to\infty}\mathbf{E}_{y-y^\gamma}^{\psi(\alpha)}\left(1_{\{\tau_y^+ \ge \tau_0^-\}}e^{-\int_0^{\tau_y^+}\varphi(v(\xi_s,y))\mathrm{d}s}\right) = 0. \tag{5.7}$$

Then we get

$$\lim_{y\to\infty}A_2(y) = C_*(\alpha). \tag{5.8}$$

Combining (5.2), (5.5) and (5.8) gives that

$$\lim_{y \to \infty} e^{y\psi(\alpha)} v(x, y) = C_*(\alpha)\Psi'(\psi(\alpha))W^{(\alpha)}(x),$$

which gives the desired result. Now we are left to prove (5.6) and (5.7). By Lemma 2.2 and Theorem 2.1, we have

$$\mathbf{E}_{y-y^\gamma}^{\psi(\alpha)}\left(1_{\{\tau_y^+ \geq \tau_0^-\}}e^{-\int_0^{\tau_y^+}\varphi(v(\xi_s,y))ds}\right) \leq \mathbf{P}_{y-y^\gamma}^{\psi(\alpha)}\left(\tau_y^+ \geq \tau_0^-\right) = 1 - \mathbf{P}_{y-y^\gamma}^{\psi(\alpha)}\left(\tau_y^+ < \tau_0^-\right)$$

$$= 1 - e^{y^\gamma\psi(\alpha)}\mathbf{E}_{y-y^\gamma}\left(e^{-\alpha\tau_y^+}1_{\{\tau_y^+<\tau_0^-\}}\right) = 1 - e^{y^\gamma\psi(\alpha)}\frac{W^{(\alpha)}(y-y^\gamma)}{W^{(\alpha)}(y)}$$

which tends to 0 as $y \to \infty$ by (2.1). Thus (5.7) is valid. To prove (5.6), for any $y > 0$, define

$$G(y) := \mathbf{E}_{y-y^\gamma}^{\psi(\alpha)}\left(e^{-\int_0^{\tau_y^+}\varphi(v(\xi_s,y))ds}\right).$$

For any $z > y$, by the translation invariance and the strong Markov property of $\xi$, we have

$$G(z) = \mathbf{E}_{z-z^\gamma}^{\psi(\alpha)}\left(e^{-\int_0^{\tau_z^+}\varphi(v(\xi_s,z))ds}\right) = \mathbf{E}_0^{\psi(\alpha)}\left(e^{-\int_0^{\tau_{z^\gamma}^+}\varphi(v(\xi_s+z-z^\gamma,z))ds}\right)$$

$$= \mathbf{E}_0^{\psi(\alpha)}\left(e^{-\int_0^{\tau_{z^\gamma-y^\gamma}^+}\varphi(v(\xi_s+z-z^\gamma,z))ds}\right)\mathbf{E}_{z^\gamma-y^\gamma}^{\psi(\alpha)}\left(e^{-\int_0^{\tau_{z^\gamma}^+}\varphi(v(\xi_s+z-z^\gamma,z))ds}\right),$$

where the first term of the above display is dominated by 1 from above and the second term is equal to $\mathbf{E}_0^{\psi(\alpha)}\left(e^{-\int_0^{\tau_{y^\gamma}^+}\varphi(v(\xi_s+z-y^\gamma,z))ds}\right)$.
It follows that

$$G(z) \leq \mathbf{E}_0^{\psi(\alpha)}\left(e^{-\int_0^{\tau_{y^\gamma}^+}\varphi(v(\xi_s+z-y+y-y^\gamma,z-y+y))ds}\right). \tag{5.9}$$

Note that for any $w > 0$, it holds that

$$v(x+w, y+w) = \mathbb{P}_{x+w}\left(\exists\, t > 0, u \in N_t \text{ s.t. } \min_{s \leq t} X_u(s) > 0, X_u(t) > y+w\right)$$

$$\geq \mathbb{P}_{x+w}\left(\exists\, t > 0, u \in N_t \text{ s.t. } \min_{s \leq t} X_u(s) > w, X_u(t) > y+w\right) = v(x, y).$$

This combined with (5.9) gives that for $z > y$,

$$G(z) \leq \mathbf{E}_0^{\psi(\alpha)}\left(e^{-\int_0^{\tau_{y^\gamma}^+}\varphi(v(\xi_s+y-y^\gamma,y))ds}\right) = G(y).$$

Thus, the limit $C_*(\alpha) := \lim_{y \to \infty} G(y)$ exists. It is obvious that $C_*(\alpha) \leq 1$. Next, we only need to show $C_*(\alpha) > 0$. We assume without loss of generality that $y$ is an integer. By the strong Markov property and Jensen's inequality,

$$G(y) = \frac{\mathbf{E}_0^{\psi(\alpha)}\left(e^{-\int_0^{\tau_y^+}\varphi(v(\xi_s,y))ds}\right)}{\mathbf{E}_0^{\psi(\alpha)}\left(e^{-\int_0^{\tau_{y-y^\gamma}^+}\varphi(v(\xi_s,y))ds}\right)} \geq \mathbf{E}_0^{\psi(\alpha)}\left(e^{-\int_0^{\tau_y^+}\varphi(v(\xi_s,y))ds}\right) \geq \exp\left\{-\sum_{n=1}^y \mathbf{E}_0^{\psi(\alpha)}\left(\int_{\tau_{n-1}^+}^{\tau_n^+}\varphi(v(\xi_s,y))ds\right)\right\}.$$

By (5.1), we get

$$\int_{\tau_{n-1}^+}^{\tau_n^+}\varphi(v(\xi_s,y))ds \leq (\tau_n^+ - \tau_{n-1}^+)\varphi(v(n,y)) \leq (\tau_n^+ - \tau_{n-1}^+)\varphi\left(e^{(n-y)\psi(\alpha)}\right).$$

Note that under $\mathbf{P}_0^{\psi(\alpha)}$, $\{\tau_n^+ - \tau_{n-1}^+\}$ are i.i.d. random variables with finite first moment. Therefore,

$$G(y) \geq \exp\left\{-\sum_{n=1}^y \varphi\left(e^{(n-y)\psi(\alpha)}\right)\mathbf{E}_0^{\psi(\alpha)}\left((\tau_n^+ - \tau_{n-1}^+)\right)\right\}$$

$$= \exp\left\{-\mathbf{E}_0^{\psi(\alpha)}(\tau_1^+)\sum_{n=0}^{y-1}\varphi\left(e^{-n\psi(\alpha)}\right)\right\} \geq \exp\left\{-\mathbf{E}_0^{\psi(\alpha)}(\tau_1^+)\sum_{n=0}^{\infty}\varphi\left(e^{-n\psi(\alpha)}\right)\right\},$$

which implies that

$$C_*(\alpha) \geq \exp\left\{-\mathbf{E}_0^{\psi(\alpha)}(\tau_1^+)\sum_{n=0}^{\infty}\varphi\left(e^{-n\psi(\alpha)}\right)\right\}.$$

According to (1.3), we have

$$\sum_{n=0}^{\infty} \varphi\left(e^{-n\psi(\alpha)}\right) \le \varphi(1) + \int_0^{\infty} \varphi\left(e^{-z\psi(\alpha)}\right)\mathrm{d}z < \infty,$$

which implies that $C_*(\alpha) > 0$. This gives the desired result. □

## Funding

## Acknowledgment

## References

[1] H. Hou, Y.-X. Ren, R. Song, Tails of extinction time and maximal displacement for critical branching killed Lévy process. Potential Anal., 57(2025) 1811–1867.
[2] H. Hou, Y.-X. Ren, R. Song, Y. Zhu, Asymptotic behaviors of subcritical branching killed Brownian motion with drift, Adv. Appl. Probab. 57(4) 1484–1509.
[3] S. Sawyer, J. Fleischman, Maximum geographic range of a mutant allele considered as a subtype of a brownian branching random field, PNAS 76 (1979) 872–875.
[4] S. Lalley, Y. Shao, On the maximal displacement of critical branching random walk, Probab. Theory Rel. Fields 162 (2015) 71–96.
[5] H. Hou, Y. Jiang, Y.-X. Ren, R. Song, Tail probability of maximal displacement in critical branching Levy process with stable branching, Bernoulli 31 (2025) 630–648.
[6] C. Profeta, Maximal displacement of spectrally negative branching Lévy processes, Bernoulli 30 (2024) 961–982.
[7] E. Neuman, X. Zheng, On the maximal displacement of subcritical branching random walks, Probab. Theory Rel. Fields 167 (2017) 1137–1164.
[8] S. Asmussen, H. Hering, Branching Processes, Boston, MA, Birkhaüser Boston, Inc, 1983.
[9] L. Chaumont, R.A. Doney, On Lévy processes conditioned to stay positive, Electron. J. Probab. 10 (2005) 948–961.
[10] M.L. Silverstein, Classification of coharmonic and coinvariant functions for a Lévy process, Ann. Probab. 8 (1980) 539–575.
[11] I. Grama, H. Xiao, Conditioned local limit theorems for random walks on the real line, Ann. Inst. H. Poincaré Probab. Statist. 61 (1) (2025) 403–456.
[12] A.E. Kyprianou, Fluctuations of Lévy Processes with Applications. Introductory lectures, Heidelberg, 2nd ed., Heidelberg, 2014.
[13] B.A. Surya, Evaluating scale functions of spectrally negative Lévy processes, J. Appl. Probab. 45 (2008) 135–149.
[14] K.L. Chung, J.B. Walsh, Markov Processes, Brownian Motion, and Time Symmetry, New York, Springer, 2nd ed., 2005.
[15] J. Bertoin, Lévy Processes. Cambridge Tracts in Math, 121, Cambridge University Press, Cambridge,
[16] I. Grama, H. Xiao, Conditioned local limit theorems for random walks on the real line, 2021, arxiv:2110.05123
[17] W. Feller, An Introduction to Probability Theory and Its Applications, 2, 2nd ed., New York, John Wiley and Sons, 1971.
[18] E. Aïdékon, Convergence in law of the minimum of a branching random walk, Ann. Probab. 41 (2013) 1362–1426.
[19] Z. Li, Measure Valued Branching Markov Processes, 2nd ed., Springer, Berlin, 2022.
[20] K. Hinderer, Remarks on directly Riemann integrable functions, Math. Nachr. 130 (1987) 225–230.
[21] E.B. Dynkin, Branching exit Markov systems and superprocesses, Ann. Probab. 29 (2001) 1833–1858.